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UNIVERSITY OF ALBERTA

# UNDERGRADUATE CALCULUS STUDENTS' LANGUAGE USE AND SOURCES OF CONVICTION

by

Sandra D. Frid

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF SECONDARY EDUCATION

EDMONTON, ALBERTA

SPRING 1992

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Example 3, page 143 (the statement of the problem, but not the solution)

Exercise 10, page 139
Exercise 32, page 140
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# ABSTRACT

The purpose of this study was to investigate student learning in introductory, undergraduate calculus from a constructivist perspective. Students' language use and sources of conviction were the focus of analysis. Sources of conviction refer to how one determines mathematical truth and validity. Three undergraduate calculus classes were involved, with students taught by one of three instructional approaches: technique-oriented, concepts-first and infinitesimal instruction. Interviews with 17 students provided data on students' language use and sources of conviction. Instructor interviews, classroom observations and textbook analyses provided description of each instructional approach.

Student interview data revealed the existence of three groups of students who differed in their sources of conviction. These groups were named Collectors, Technicians, and Connectors. Collectors exhibited the highest degree of external sources of conviction, using teacher or textbook presentations as means by which to determine truth or validity. Their calculus conceptualizations were constructed as a collection of isolated, relatively unconnected statements, rules and procedures. Technicians based truth and validity upon their knowledge of the logical, organized structure of calculus, and constructed their conceptualizations as a logical organization of statements, rules and procedures. Connectors exhibited the highest degree of internal sources of conviction, displaying a sense of personal understanding of calculus. Their conceptualizations were constructed as a network of connections between various aspects of calculus and between calculus and themselves.

Analysis of students' language use indicated they used pre-calculus and everyday language, visually and physically oriented language, and procedural language knowledge in their calculus responses. Except for Connectors, students did not make extensive use of symbols to interpret or explain calculus ideas. Students who received infinitesimal instruction used infinitesimal language related to notions of infinitesimal closeness and infinite magnification of a curve.

The study revealed the existence and characteristics of three types of calculus learners, and there was some relationship between these groups and competency in calculus. The study also revealed that students used language as a source of conviction. Finally, students' perceptions of their own learning were revealed as an unexpected but important element in the nature of their sources of conviction and construction of calculus conceptualizations.

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# 1. INTRODUCTION AND STATEMENT OF THE PROBLEM

## A. Introduction

Over the past decade constructivism has emerged as an important influence in mathematics education research. This is because constructivism has been proving to be a valuable perspective from which to understand mathematics learning (Ernest, 1989). Constructivism provides a model of knowledge which views mathematics learning as an individual, evolutionary process. This is in contrast to a view of learning that sees concepts as transferrable, "ready-made" from teachers to learners. That is, constructivism views mathematics learning as an active, constructive process (Schuell, 1985) in which an individual builds up knowledge for himself or herself.

Ernest (1991) discusses the above notions through discussion of mathematics as a social construction. This philosophy, known as social constructivism, takes the view that "human language, rules and agreement play a key role in establishing and justifying the truths of mathematics" (p.42). That is, mathematical knowledge is grounded in the following: (1) "linguistic knowledge, conventions and rules" (p.42), (2) social processes by which an individual's internal, subjective knowledge is turned into external, objective knowledge, and (3) objectivity viewed as public, social acceptance rather than an inherent property of the content of knowledge. These features imply that mathematical knowledge is dependent upon social sharing of language and decisions pertaining to truth and validity. Further, as a consequence for mathematics education researchers, these points imply language use and ways of determining truth and validity are likely to be important components of mathematics learning.

Constructivist notions are prevalent in the ideas and research of individuals involved in mathematics education research, even when these individuals do not explicitly state the psychological bases from which they work. Skemp (1987) clarifies a prime reason for this. He notes that models of learning which see learning as a passive, reproductive process have been unsuccessful in both explaining and bringing about "the higher forms of learning . . . of which mathematics is a clear example" (p.134). A constructivist model of learning is being widely accepted by today's mathematics education researchers because it gives insights into poorly understood learning phenomena such as: why students learn different things from the same instruction, why many students do not appear to learn by being given information and explanations, and why students appear to develop misconceptions of concepts. These reasons indicate research from a constructivist perspective might be particularly useful in studying student learning in calculus.

Few studies have been done to examine student learning in calculus, and many that have been done have focused on student errors, misconceptions, or inability to perform certain tasks. For example, Seldon et al. (1989) found that in a sample of 17 students obtaining C's in an introductory calculus course, no one was able to solve problems for which they had not been taught a method of solution. Davis and Vinner (1986), in an investigation of high school students with a full year of calculus study, found students held numerous misconceptions about both informal and formal ideas related to limits.

In an earlier study based on clinical interviews with 110 calculus students of ages 16 to 22, Orton (1983a, 1983b) investigated students' understandings of limits, differentiation, and integration. Interview responses were analyzed in terms of students' conceptual, algebraic, and (apparently) arbitrary errors and misconceptions. Orton found that students made numerous errors in both intuitive and formal aspects of concepts. For example, some students failed to recognize the connection between the exact area under a curve and the use of approximating rectangles, some saw rotating secants as disappearing to a point rather than becoming a tangent line, and others treated $\infty$ as an algebraic symbol which could be manipulated in the same way as numerals.

These investigations have given insight into students' misunderstandings in calculus. What is needed now is research into how instruction can better guide and support student learning in calculus. This is particularly important when one considers that the drop-out and failure rates in calculus are high compared to other undergraduate courses. Figures between 30% and 50% are reported in the literature (Peterson, 1987; Cipra, 1988). Even students passing a calculus course tend to perform at low levels with respect to both skills and the use of calculus ideas (Peterson, 1987; Cipra, 1988).

This is an unfortunate state of affairs in that many undergraduate students must be successful in an undergraduate calculus course to either begin or continue enrollment in various programs. Calculus is a required course for these students if they are aiming for careers in science, engineering, medicine, business, education, and various other fields. Although reasons for the inclusion of calculus in these curricula will not be discussed here, it can be argued that, beyond the knowledge and skills imparted, calculus acts as a "filter" for entry into numerous careers.

Some educators across North America say the situation as outlined is indicative of a crisis in the teaching of calculus (Peterson, 1986, 1987; Cipra, 1988). Over the past few years this has led to much support for change in introductory calculus courses. Educators in favour of change have been developing and implementing changes. These changes include the following: (see Douglas, 1986; Peterson, 1986) (1) shifting the focus of calculus teaching to the fundamental ideas of calculus, rather than emphasizing drill in routine skills and techniques, (2) integrating applications into the body of calculus courses by reinforcing the role of approximations and problem situations with contexts relevant beyond the field of mathematics, and (3) producing textbooks to support curriculum changes.

For improved success of student learning in calculus more research is needed into various instructional emphases and formats and their subsequent effects on learning. Further, since constructivism shows promise as a useful means by which to study calculus learning, research from this perspective is indicated. This in turn points to a research focus on language use and the ways that truth and validity are determined.

## B. Statement of the Problem

The research objective of this study is the investigation of student learning in calculus from the point of view of constructivism. Students taught by one of three approaches to calculus are the focus of the study, and the main question addressed is the following:

In what ways do calculus students' *language use* and *sources of conviction* reflect the nature of their constructivist learning?

More specifically, the study analyzes: (1) the nature and role of the language students use to interpret calculus concepts and problems, (2) the nature and role of students' convictions regarding the validity or truth of calculus interpretations and problem responses, (3) the ways students construct their calculus conceptualizations, and (4) the ways different approaches to calculus instruction impact on students' *language use, sources of conviction*, and manner of construction of conceptualizations. As a subsidiary, necessary component of addressing this fourth area of inquiry the following is also addressed: (5) the ways three different approaches to calculus instruction translate into classroom, textbook and exercise assignment instructional events. That is, a necessary component of this study is description of the nature of the three approaches to calculus instruction as delivered to students.

## C. Definition of Terms

*Students* in this study are college and university undergraduates enrolled in an introductory calculus course.

*Calculus concepts* refer to the limit and the derivative.

*Instruction* refers to teaching that takes place in the classroom or laboratory, or in interaction with a textbook or curricular materials.

*Calculus problems* are mathematical situations (concrete or symbolic in context) that a learner is required to explain or act upon. This would be through one or more of interpretation, construction, or symbolic manipulation.

*Conceptualizations* are mental representations an individual has for concepts. These are, according to a constructivist perspective, dynamic and individually constructed internal mental images, properties, and processes (von Glasersfeld, 1987a).

*Constructivist learning* refers to the building of conceptualizations as a result of experience. More specifically, constructivist learning refers to learning viewed as an individual process in which a learner constructs knowledge for herself or himself. It is learning seen as an individual process of making sense of new information by relating it to and reorganizing conceptual structures and process (Kilpatrick, 1987; von Glasersfeld, 1987a).

*Language use* includes use of verbal and written words and symbols, and is categorized as either *technical* or *everyday language* use.

*Technical language* refers to mathematical words and symbols accepted as proper and correct by the mathematics community at large.

*Everyday language* includes words and symbols which are not recognized by the mathematics community for use in unambiguous mathematical discourse. These words and symbols might or might not be mathematical in nature, and are often found in daily English *language use.* Also included in this language category are *pre-calculus language* and *visually oriented language*. *Pre-calculus language* refers to words that are part of an individual's language knowledge prior to studying calculus. For example, knowledge of terminology such as "limit" or "continuous" is pre-calculus language knowledge. *Visually oriented language* refers to words whose use is in the context of visual, physical interpretations or representations of concepts. Examples include "slope", "increase" "flat", "hole" and "straight".

*Procedural language use* refers to use of *technical* or *everyday language* as means to describe mathematical operations or procedures.

*Sources of conviction* refer to where an individual sees truth and validity residing in the context of learning calculus. That is, *sources of conviction* refer to how individuals determine facts and accordance with accepted mathematical principles and standards, and how individuals determine legitimacy, consistency and logicality. Responsibility for the determination of truth or validity could lie within various sources. These include: the teacher's knowledge, the statements of a textbook or other instructional materials, the inherent physical structure of the world, a student's knowledge of the structure and rules of mathematics, or a student's own personal beliefs.

## D. Significance of the Study

This study is important because it focuses upon three important implications arising from constructivism. These implications arise from the way constructivism conceptualizes the teaching and learning of mathematics (von Glasersfeld, 1987a): (1) communication does not occur via words which serve as "containers" to transfer meaning, but rather, communication involves subjective interpretations of language (p.7), (2) a student's learning should be viewed as "successful organization of her or his own experience", rather than "replication of what the teacher does" (p.6), and (3) "making sense" out of instruction "means finding a way of fitting available conceptual elements into a pattern that is circumscribed by specific constraints" (p.9). These three points are in agreement with constructivist notions as discussed by Ernest (1991) and outlined in Section A of this chapter.

The first of these three consequences implies that an important feature of understanding student learning in calculus is understanding how students interpret and use language related to calculus. Consequence (2) reveals a need to better understand how calculus students attribute truth and validity in the context of learning calculus. That is, understanding the ways in which students establish "correctness" is important to understanding student learning because these *sources of conviction* reflect the nature of what is learned. Consequence (3) points to a need to consider the personal and situational conditions within which an individual's learning occurs. More specifically, since constructivism sees learning as a process of adaptation to one's experiential world, the instructional conditions of this world should be considered as important influences upon learning. Thus, this study examines in a range of environments the usefulness of ***language use*** and *sources of conviction* as elements of constructivist learning in calculus. These constructivist related notions are therefore formulated and refined by this study. In addition, since the research studies three different instructional settings, it tests the usefulness of a constructivist perspective in studying calculus learning in a range of environments. In this way constructivist notions are refined and clarified, as are techniques for studying student learning from a constructivist perspective.

As a result, outcomes of this study have theoretical and practical impact in four areas: (1) development of methods for studying calculus instruction and student learning, (2) development of a better understanding of calculus instruction and student learning in calculus, (3) theory and related methods for facilitating calculus learning as a meaningful endeavour, and (4) theory and related methods for incorporating constructivism into mathematics education research.

### E. Description of the Study

The following outline is an overview of the study. A more extensive presentation and discussion appear in Chapter 3.

Clinical and personal interviews with 17 students were the primary method of inquiry into the nature and role of student's *language use*, *sources of conviction*, and manner of construction of calculus conceptualizations. These were done in the last three weeks of the school term. Problems used for the clinical portion of the interviews were selected from pilot study questions on the basis of information gathering capacity.

The research is a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions. These include a large university and two small private colleges. The course at the university is representative of introductory calculus courses across North America in its content and an emphasis upon learning techniques for differentiation, integration, graphing, and problem solving. In comparison, one of the colleges uses a "concepts-first" approach to instruction in which concepts are explored intuitively before introduction of their formal definitions and proofs, and before skill development is emphasized. The second college uses an instructional approach which develops concepts intuitively while using infinitesimal methods related to nonstandard analysis. These infinitesimal methods are the tools by which Newton and Leibniz first developed calculus. This range of instructional settings allowed examination of the impact of different approaches to instruction on the nature and role of students' *language use*, *sources of conviction* and manner of construction of conceptualizations.

Classroom observations were done for a 13 week school term and each class was observed for 25% to 50% of regular classroom time. Along with textbook and exercise assignment analyses the observations provided a description of each instructional setting in terms of *language use* and the ways truth and validity (*sources of conviction*) were determined. Additional background information for each of the classes was obtained from student questionnaires given at the beginning and end of term, and from instructor interviews. These activities also served to provide data by which the impact of instruction on students' *language use*, *sources of conviction* and manner of construction of conceptualizations could be studied.

## F. Delimitations

1. Analysis of the clinical and personal interviews with students focused on students' *language use*, *sources of conviction* and manner of construction of conceptualizations.

2. A small number of students, 5 or 6 from each of the three classes, were interviewed. A small number allowed opportunity for extensive responses from students on both the clinical and personal interview questions.

3. Student interviews were delimited to one main 1 to 2 hour interview and a short 15 to 30 minute follow-up interview. Responses to the calculus problems in the clinical portion of the interviews were made in the presence of the researcher.

4. Not all main concepts in introductory calculus were focused on. The limit and the derivative concepts were focused on, while the integral was not. This delimitation was because in each of the three calculus classes integration was taught at the end of school term, after some of the interviews were scheduled to take place.

5. The duration of the classroom observations was one 13 week school term. These observations, as well as the textbook and exercise assignment analyses focused on the *language uses* and *sources of conviction* that were displayed.

## G. Assumptions

1. There is a relationship between students' *language use* and *sources of conviction*, and her or his calculus conceptualizations.

2. There is a relationship between a student's calculus conceptualizations and his or her responses to calculus related questions both orally and through writing.

## H. Limitations

1. The students interviewed were not randomly selected. They were volunteers, but they represented a range of backgrounds and achievement levels in their respective calculus courses.

2. Since students were required to respond to problems in the presence of the researcher, the researcher's presence affected the manner in which students responded. Students might have responded differently if no one or a different individual had been present, and their responses would not have been as verbal in nature if they had been required to respond to problems without another individual present.

3. Since the concept of the integral was not present in the interview problems, not all aspects of student learning of introductory calculus were examined.

4. Students were required to respond to a small number of calculus problems only, so not all aspects of their calculus *language use, sources of conviction* and manner of construction of conceptualizations could be examined. That is, interviews cannot reveal all aspects of the construction, nature, and adaptation of an individual's conceptualizations. However, the calculus problems were selected to provide a range of calculus tasks related to the limit and derivative concepts.

5. The classes were not randomly chosen. They were selected to provide a range of instructional settings, and opportunity for examination of the impact of different approaches to instruction on students' *language use, sources of conviction* and manner of construction of conceptualizations.

6. Classroom observations were not conducted in every class taught. Since they were carried out in 25% to 50% of each instructor's classes, not all instructional events were observed.

## I. Outline of the Report

Chapter 2 contains a review of selected relevant literature. A detailed account of the design of the study comprises Chapter 3. This includes the pilot study, interview and questionnaire items and their rationale, and classroom observation, textbook and exercise assignment, and interview research methods and analysis procedures. In Chapter 4 the results of the study are reported, including background profiles for each class, instructor interview summaries, classroom observation and textbook and exercise assignment analysis findings, and student interview results. Chapter 5 contains a summary and discussion of the findings with reference to the specific questions posed in the statement of the problem. In addition, a discussion of some of the educational implications of the findings and suggestions for further research are given in this final chapter.

# 2. REVIEW OF RELATED LITERATURE

(Quotations and examples in this chapter that are not referenced are taken from pilot study work done in May and June, 1990).

## A. Introduction

This study is an investigation of student learning in calculus. It adopts a constructivist perspective of learning and focuses upon the role *language use* and *sources of conviction* play in learning.

This chapter begins with a presentation of the critical notions underlying constructivism and discusses their implications for mathematics education and mathematics education research. In particular, theory related to *language use* and *sources of conviction* is presented. Following this is an overview of research related to student learning in calculus. Generally these studies have not explicitly adopted a constructivist perspective. However, implications of the results of these studies in terms of constructivism are outlined.

Background to the research methodology is then presented. The final section summarizes key aspects and links between the various theoretical and research areas discussed previously.

## B. Constructivism

Traditionally, school mathematics has been concerned with the transmission of knowledge from teacher to student. Related to this model of education is Platonism, the idea that what is learned exists independently of participants in the learning process (Hoyles, 1985; von Glasersfeld, 1987a). In such a context learning mathematics is seen as a passive, reproductive process. At present many mathematics education researchers consider learning mathematics to be an active, constructive process (Schuell, 1985). Skemp (1987) clarifies reasons for this view. He notes that models of learning which see learning as a passive, reproductive process have been unsuccessful in both explaining and bringing about "the higher forms of learning . . . of which mathematics is a clear example" (p.134). The result has been that a theory of knowledge known as constructivism has emerged as an important influence in mathematics education research. Constructivist notions are prevalent in the ideas and research of individuals involved in this field, even when these individuals do not explicitly state the psychological bases from which they work.

Piaget is considered by many to be a pioneer in the formulation of constructivist views (Narode, 1987). His analyses of the intellectual development of children resulted from asking the epistemological questions: "What do we know? How do we know it?" In particular, two major assumptions underlying constructivism are in accordance with Piagetian theory. These assumptions are: (1) learners actively construct knowledge, and (2) an individual's prior knowledge plays a critical role in learning and performance (Putnam, 1987).

Thus, according to a constructivist perspective, concepts cannot be transferred "ready-made" from teachers to learners. Rather, conceptual development is seen as an individual process in which a learner constructs or builds up knowledge for himself or herself. This is viewed as being done piece-by-piece from available elements in the individual's physical and mental environment (Schuell, 1985; Skemp, 1987; von Glasersfeld, 1987a). More specifically, through a process of abstraction (becoming aware of similarities between experiences), an individual gathers together regularities in experiences to form what is referred to in Piagetian terms as schemata (Thomas, 1979). Schemata are internal conceptual networks, hierarchies, and processes. At a primary level, constructivism sees these conceptual entities as derived from sensory and motor experiences. Further development occurs when they interact with additional sensory and motor experiences, and with each other (Skemp, 1987).

In a constructivist orientation, the importance of an individual's prior knowledge emphasizes that not only do learners construct their own knowledge, they do so by relating new information to what has previously been learned. That is, learners make sense of new information by connecting it to and reorganizing their conceptual structures and processes. In Piagetian terms this would correspond to the processes of assimilation and accommodation (Thomas, 1979). Therefore, constructivism sees an individual's existing conceptual structures and processes as a major determinant in how she or he interprets and comprehends experiences.

In particular, constructivism sees learning as an adaptive process which, through trial and error, the individual constructs a viable model of the world (von Glasersfeld, 1984). This model is seen by constructivism as a fit of knowledge to experience, rather than a match between knowledge and reality. Instead of discovery of an "independent, pre-existing world" (Kilpatrick, 1987; p.7), learning is viewed as adaptation to what is experienced within the world. That is, learning is "organizing experience so as to deal with a real world that cannot itself be known" (Kilpatrick, 1987; p.6). This is a rejection of metaphysical realism. The constructivist model of learning which rejects realism is called radical constructivism (von Glasersfeld, 1984; Kilpatrick, 1987). If the model sees knowledge as an individual construction, but does not accept rejection of realism, then it is called trivial constructivism.

Rejection of realism is a controversial aspect of constructivism. It separates constructivism from the practice of many mathematicians and mathematics teachers. These people generally accept the existence of external, objective mathematical objects and truths. That is, mathematical facts are generally seen to be "what *they* are, not what we wish them to be" (Davis & Hersh, 1981; p.362). In comparison, constructivism sees all knowledge as being subjective and individually created. Thus, it might be said that constructivism does not adequately explain common mathematical practice. Its ability to study and describe classroom practice and student learning might therefore be limited.

However, since this remains to be determined, it might be that constructivism can give new insights into these descriptions. In particular, if one views objectivity as a social construction rather than an inherent property of mathematical concepts, then constructivism is able to explain mathematical practice. According to constructivism, objectivity resides in the "public nature of language, of concepts, of theories and hence of knowledge" (Lerman, 1989; p.219). This implies objective knowledge arises from social, public negotiation and construction. This is not contradictory to either mathematics or the learning of mathematics. For example, the history of mathematics demonstrates that mathematical knowledge is not a "body of immutable and necessary truths" (Ernest, 1991; p.62). In fact, the development of non-Euclidean geometries, the work of Godel, and more recently, the rise of chaos theory, fractals, and alternative logics all provide evidence that mathematics is not immutable. These areas of mathematics also demonstrate that objective mathematical knowledge lies in the "shared rules, conventions, understandings, and meanings of the individual members of society, and in their interactions" (Ernest, 1991; p.82). An individual's learning of mathematics can therefore be seen as a process of subjective construction of publicly shared knowledge of words, symbols, rules and conventions. Thus, by attributing the term "understanding" to public practice and use of concepts, constructivism can explain mathematical practice.

This is one way language plays a key role in learning viewed from a constructivist perspective. Other aspects of the importance of language to learning will now be discussed. Although many of these ideas were formulated before the emergence of constructivism, they are relevant to understanding language use from a constructivist perspective.

## C. Language Use

Historically, the relationship of language to learning has been a focus of much literature in psychology, linguistics and education (Gatherer, 1977). Vygotsky (1962) suggests language and thought are related in that language is inextricably linked to the acquisition of concepts. He says:

> . . . the birth of a new concept is invariably foreshadowed by a more or less strained use of old linguistic material; the concept does not attain to individual and independent life until it has found a distinctive linguistic embodiment (p.74).

Bruner (1975) also views language as an instrument of thought. He emphasizes the importance of "using" language, and argues that language and thought come together through "regulation of action". Learning the meanings of words involves mastering "a set of component procedures relating to their use" (p.65). Constructivist notions are in accordance with the views of Vygotsky and Bruner in that constructivism highlights mastery of publicly shared language use, and the processes of assimilation and accommodation.

Johnson (1987) also discusses the important role language plays in learning. He explains linguistic meaning in terms of what he refers to as "image schemata". Image schemata are structures of meaning arising from "perceptual interactions and bodily movements within our environment" (p.19). Johnson's notions of image schemata go beyond the concept of schemata as mental images or pictures. According to Johnson, image schemata function on a more general and abstract level than the formation of particular images. Particular images do not fully portray a concept, whereas an image schemata is "an organized, unified whole within our experience and understanding that manifests a repeatable pattern or structure" (p.44). For example, specific images of a right-angled triangle, an obtuse-angled triangle or an acute-angled triangle do not fully portray the concept of triangle. In each individual instance the underlying triangle schemata is manifested in a different way, while a recognizable form is retained.

Examples Johnson uses to elaborate his claim that image schemata arise from bodily experiences are those of physical containment or boundedness, physical forces and physical balance. Only the example of balance will be outlined here because it is sufficient for explicating Johnson's ideas. Johnson describes how meanings of the term "balance" are connected to physical experiences of balance. For example, notions of physical, bodily experiences of balance are found in understandings of such things as: "balanced personalities, balanced views, balanced systems, balanced equations, the balance of power, the balance of justice, and so on" (p.87). The word "balance" is used in a wide variety of domains because the domains "are structurally related by the same set of underlying schemata metaphorically elaborated" (p.96). In the case of "balance" the underlying schemata is the bodily experience of equilibrium. Equilibrium therefore serves as an underlying metaphor to interpret "balance" in other domains.

Metaphor is also a focus of Pimm's discussions of the role language plays in learning (Pimm, 1987). His discussions are specifically in the context of mathematics learning. He explains how metaphors can guide the creation of meanings by an association of the less familiar with the more familiar. Two types of metaphors often found in mathematical language use are identified by Pimm: extra-mathematical and structural metaphors. Extra-mathematical metaphors are used to interpret mathematics in terms of objects or events in the physical world. Examples include: a graph is a picture, a function is a machine, and an equation is a balance. Structural metaphors involve an extension of ideas from within mathematics itself. Examples include: the notion of slope of a line extended to slope of a curve, multiplication of numbers extended to multiplication of matrices, and the notion of number extended to complex numbers.

Pimm notes that although mathematical metaphors serve as tools for thought, they have limitations. They are frequently associated with physical world interpretations that might not be consistent with the corresponding mathematical concept. Examples of these associations include: "talking of expressions *vanishing*, functions *obeying* a rule or being *well-behaved*, the *inheritance* of mathematical properties or discovering mathematical *laws*" (p.95). Subjective interpretations of these words are likely to vary greatly. For example, "vanishing" could be interpreted as disappearing, being hidden, or no longer existing. Mathematical "laws" might be interpreted as statements that have exceptions and can be broken. These notions related to "laws" or "vanishing" are not necessarily congruent with each other or with the corresponding mathematical ideas.

Kaput (1979) also emphasizes that the use of mathematical terms and symbols is filled with physical notions, and he notes that although the origin of mathematical language is frequently neglected by mathematicians and educators, it remains important in learning mathematics. Kaput cites calculus as exemplary of this claim in that primary calculus concepts have been given meaning through a collection of motion ideas. These ideas are reflected in both notation and terminology. For example, the symbol for the concept of limit is given a motion connection using an arrow:

$$\lim_{x \to a} f(x) = L$$

This is read: "as x approaches a, f(x) approaches L". Additionally, calculus is full of image-laden motion words such as converge, diverge, increasing, constant, and transformation.

Kaput notes that formalization in calculus tends to neglect the underlying ideas. The ideas are replaced by such things as rigorous $\varepsilon$-$\delta$ definitions and proofs. Kaput views this replacement as disastrous in the teaching of calculus because it neglects the process of creating connections between the formal, accepted mathematics knowledge universe and the human knowing universe. That is, it fails to acknowledge the physical world context within which calculus was constructed and is currently learned and used.

The role that symbol systems play in learning is treated by Skemp also (Skemp (1987). Skemp includes both the English language and mathematics language as symbol systems. According to Skemp, the functions of symbol systems include: communication, recording of knowledge, explanation, classification, helping to show structure, making reflective activity possible, making routine manipulations automatic, recovering information and understanding, and promoting creative mental activity. Through these functions, symbols (and therefore language) act as a "combined label and handle for identifying and manipulating concepts" (p.62). Language can therefore be said to be essential in mathematics learning. This is because language is the predominant form of human communication (Skemp, 1987), and more importantly, because it is "the means of actively centering attention, of abstracting certain traits, synthesizing them, and symbolizing them by a sign" (Vygotsky, 1962; p.81). Pimm (1987) notes that "we name things for reference, and hopefully for ease of reference, to draw attention to the thing named" (p.127). He also remarks that, since naming causes us to look at the thing named in particular ways, certain attributes are stressed while others are neglected.

Another aspect of language and learning that is important to constructivism is the role of natural, everyday language. A prime reason for this is that a term for a mathematical concept that also exists in natural, everyday language carries with it a whole set of natural language meanings (Halliday, 1978). Natural language meanings are therefore important factors in learning. Constructivism also highlights the importance of natural, everyday language meanings by stressing previous experiences and understandings as important to construction of conceptualizations.

The role that everyday English language plays in mathematics learning is dealt with extensively by Pimm (1987). Pimm uses Halliday's notions of a register. A register is "a set of meanings appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1975; p.65). A register is therefore "not just use of technical terms" (Pimm, 1987; p.76). It also involves certain phrasing and characteristic modes of arguing. Pimm notes that the mathematics register contains a mixture of specialist mathematics terms and terms "borrowed" from everyday English. Difficulties in learning can result from this because "borrowed" terms can be associated with non-mathematical meanings. Examples include: "*face, degree, relation, power, radical, complete, integrate, legs, product, moment, mean, real, imaginary, rational* and *natural*" (p.78). Pimm states that non-mathematical meanings can cause confusion and misunderstanding because they are not always in agreement with the precise mathematical concept. Halliday (1978) also supports this conclusion and observes that everyday language is not clear-cut and precise. "It is a human creation and therefore inherently messy" (p.201).

Ernest (1991) outlines how the use of a natural language such as English implicitly involves mathematical meanings, rules and conventions. He states:

> Natural language includes the basis of mathematics through its register of elementary mathematical terms, through everyday knowledge and the uses and interconnections of these terms, and through the rules and conventions which provide the foundation for logic and logical truth (p.75).

For example, terms such as "circle", "one", "two", "add", "less", and "greater" can be directly related to individuals' shared worlds of experiences. Ernest argues that language use underlies mathematics learning because it is through shared language use that "individuals construct subjective theories or personal representations" (p.72) of the concepts encountered in language interactions. On a more general level, an individual's "acquisition of language involves the exchange of utterances with other individuals in shared social and physical contexts" (p.71).

As with the ideas of Pimm and Halliday, Ernest's arguments indicate that an individual's use and interpretation of everyday language are likely to figure prominently in that individual's mathematics learning. The use of everyday language terms, as well as non-mathematical meanings ascribed to specialized mathematical language, should therefore be seen by mathematics educators as important aspects of the teaching and learning of mathematics.

To summarize, the literature on language discusses how language serves as an intermediary between sensory-motor and internalized, mental experiences. In this way language is seen as an essential component of the building of meaning from experiences. It is "the means of actively centering attention, of abstracting certain traits, synthesizing them, and symbolizing them by a sign" (Vygotsky, 1962; p.81). Thus, the literature suggests that it is to a large extent through language that an individual constructs conceptualizations.

### D. *Sources of Conviction*

Constructivism implies an individual's learning should be viewed as "successful organization of his or her own experiences" (von Glasersfeld, 1987a; p.6), rather than "replication of what the teacher does" (p.6). How students attribute truth and validity is therefore important to understanding their learning. The nature and origin of what is learned will be reflected in the *sources of conviction* a student uses to establish "correctness".

West and Pines (1985) discuss a constructivist view of learning while focusing upon knowledge sources. According to West and Pines learning is a process of making sense of inputs. It involves interaction between a learner's present understandings and whatever the new inputs are. West and Pines identify two sources of an individual's knowledge. Following Vygotsky (1962), they call one source intuitive knowledge. This is knowledge resulting from interaction with the environment. It is acquired over time and without a particular direction. One of its main characteristics is that it constitutes an individual's personal reality. An example used by West and Pines is that of learning the nature of the Earth: "We know that the earth is flat to our eye, yet round from photographs of space. We know about satellites, the shuttle, and a whole gamut of other things" (p.3). That is, knowledge of the nature of the Earth results from an individual's encounters over time with his or her physical and social worlds. The other source of knowledge West and Pines outline they refer to as formal or school knowledge. They describe it as follows:

> It is someone else's interpretation of the world, someone else's reality. Its primary characteristic is authority. It is 'correct'; it is what the book says; what the teacher says. It is approved by a whole bunch of other people who are usually older and more highly regarded than the student. Our learning of this knowledge is goal-directed. That is, we set out to learn, usually through instruction, a particular body of knowledge. We are usually expected to learn it in a certain time period. We are usually expected to demonstrate, most often through tests, what we have learnt about it (p.3).

West and Pines view learning as being comprised of the integration of intuitive and school knowledge. To explicate the learning process they introduce a vine metaphor. This metaphor uses one vine to represent intuitive knowledge of the world and another vine to represent formal or school knowledge. The extent that these two vines intertwine is seen to correspond to a variety of learning states. Four states identified by West and Pines are: (1) a conflict situation in which both vines are well established but are not in agreement, (2) a congruent situation in which intuitive and school knowledge can be integrated without problems, (3) a symbolic knowledge situation where there is little to the intuitive knowledge to interact with the school knowledge, and (4) an unrestricted situation where there is little or no school knowledge, but there is intuitive knowledge.

The vine metaphor might be adapted to serve as an indicator of a student's *sources of conviction* by classification of a *source of conviction* into one of two categories: (1) authoritative, external sources that correspond to truth and validity claims arising from formal or school knowledge, and (2) personal, internal sources that correspond to truth and validity claims arising from intuitive knowledge. However, it is not clear that formal or school knowledge sources are necessarily seen by an individual to be external to herself or himself, and intuitive knowledge sources are not necessarily viewed as internal. It might be that how a student perceives a particular knowledge source affects whether the knowledge source is external or internal in nature for that student. That is, it might be that the degree to which an individual personalizes knowledge that determines whether the knowledge source is internal or external. However, the validity or practicality for mathematics educators of this viewpoint remains to be determined.

To further explicate the role of school knowledge West and Pines note that schooling often forces students to ignore their own reality. This is because school settings tend to rely heavily on symbolic knowledge (formal use of words and symbols). A student who wants to genuinely make sense out of instruction, "as opposed to rote learning numerous isolated knowledge bits" (p.5), must therefore concentrate on constructing meaning from symbolic knowledge. Since calculus is highly symbolic and inter-related in nature, this viewpoint of the nature of schooling might give insights into students' calculus conceptualizations. However, as with the adaptation of their dichotomy to *sources of conviction*, this remains to be determined.

A similar view to the intuitive-school classification of sources of knowledge is that of private and public understandings (West et al., 1985). Private understandings are individual and arise from an individual's interpretations and internalization of public knowledge. Public knowledge is knowledge found in books, scientific papers, and other written or spoken documents. It is derived from individuals' private understandings and exists "because there is substantial overlap between the private understandings of different individuals" (p.30). From the standpoint of viewing knowledge as either private or public, learning is seen as a process of giving personal meaning to public knowledge.

In terms reminiscent of private and public understandings, Ernest (1991) discusses what he refers to as subjective and objective knowledge. According to Ernest, subjective knowledge arises from an individual's world of conscious experiences, while objective knowledge centres on products of the human mind such as formal and informal theories, proofs, and related discussions. Objective knowledge also includes any implicit knowledge contained in publicly shared, intersubjective knowledge. An example of such implicit knowledge is the shared rules and conventions of language use discussed in the previous section of this chapter.

As with the vine metaphor, the notions of subjective versus objective, or private versus public knowledge show promise as means to clarify and refine the concept of *sources of conviction* as it is conceived in this study. However, a simple dichotomy cannot adequately explain *sources of conviction* unless it acknowledges that an individual's determination of truth or validity can lie within various sources, including: the teacher's knowledge, the statements of a textbook or other instructional materials, the inherent structure of the world, a student's knowledge of the structure and rules of mathematics, or a student's own beliefs. For example, a student might give one or more of the following statements as justification for why the graph of a function is or is not continuous:

> Cause that's the way I've been taught.
>
> But we have to look at the whole picture, so you could say that the whole thing is discontinuous because of the break .
>
> By the table of values, yes.
>
> And then if it's infinitesimally close on either side, so that it all rounds off to the same number, then the function must be continuous.
>
> All polynomials are continuous.
>
> Because it's continuous if ah, if there are no jumps in it. You do not have to lift your pencil.

These examples from the pilot study demonstrate that any classification of *sources of conviction* must include a range of possibilities.

The internal and external classification of *sources of conviction* as initially conceived in this study addressed this issue of a need for a range of possibilities (see Chapter 3). In addition, *sources of conviction* were initially conceived in the following way as an implication of constructivism. Constructivism implies that student learning depends upon "recognition and re-construction of problems as being their [one's] own" (Balacheff, 1990; p.259). That is, the nature of an individual's calculus conceptualizations will be influenced by how the individual attributes truth and validity. For example, a student who finds limits of indeterminate forms by following a set of "rules" because "they work" or "that is what the teacher taught", is using an external *source of conviction.* In contrast, a student who uses the same rules but tries to make sense of them in terms of why they are used and why they work sees himself or herself as a source of the determination of truth. The student's source of knowledge arises internally, from within herself or himself. A student might also combine *sources of conviction*, making use of his or her own ideas alongside those of some other source. This can be seen in the following words of a student explaining why no derivative exists at a point of discontinuity on a graph:

> It just comes down to a single point as opposed to a point, say here at B, which does have a derivative. It doesn't come to a single point, . . . it flattens off at the top. So you can take the slope there, but you can't take it there. Or as he [the teacher] would say, you can scratch your back there. . . . I didn't totally get that scratch your back part. That's sort of a weak explanation I guess.

This student began his explanation with a description of how he saw the situation. Only afterwards did he bring in justification in terms of what the teacher might say. What is also of interest here is that this student is aware that both his and the teacher's explanations are "weak" rather than precise or conclusive.

It must be noted that students are not necessarily aware of their *sources of conviction*. In particular, if truth and validity are seen by a student to reside in such things as teacher or textbook statements, or the student's own beliefs, then the student might not be aware of either a need for or means of more formal justification. That is, students might be willing to make mathematical statements without regard for their truth or validity. If questioned on these statements they might or might not see a need for, or be able to give justification. This is reflected in the following portion of an interview with a student:

> Student: There's no tangent line to this point here, so you can't take the slope of it.
>
> Interviewer: And why is there no tangent line?
>
> Student: Um. I don't think it would be defined right at that point.
>
> Interviewer: Do you know why you think that?
>
> Student: Intuition?
>
> Interviewer: Can you put words to your intuition?
>
> Student: Words to my intuition? Not really. No. Just that I don't think you can draw a tangent line at that point. And if you can't take the slope of it, then it's undefined there [referring to the derivative].

Another important point to be noted is that a student's *sources of conviction* and resulting conceptualizations will be influenced by the conditions surrounding her or his efforts to learn. For example, if a student is motivated by "getting correct answers" his or her convictions are likely to originate from the teacher, textbook, or other external source. However, it must be noted that this could be the reality for many students. The conditions within which students make sense out of calculus instruction will therefore play an important role in what they learn.

## E. The Teaching and Learning of Calculus

Research into the teaching and learning of calculus has not been extensive. There have been studies related to student achievement in calculus, but they have been limited in number (for example, see Hirsch et al., 1983; Edge & Friedberg, 1984; Seldon et al., 1989). Further, although there have been reports and discussions of the general state of calculus instruction, related areas are in need of study (Douglas, 1986; Peterson, 1986, 1987; Cipra, 1988). The situation for research into students' conceptual learning in calculus is similar. The studies done have most frequently centred on student understandings of limits (for example, see Tall & Vinner, 1981; Davis & Vinner, 1986; Sierpinska, 1987; Williams, 1991). Student understandings of differentiation or integration have been investigated to a lesser extent. In addition, studies of student learning that have been done have focused largely on student errors, misconceptions, or inability to perform certain tasks (for example, see Seldon et al., 1989; Davis & Vinner, 1986; Orton, 1983a, 1983b; Williams, 1991).

Thus, there remain many unanswered questions regarding the teaching and learning of calculus. A number of these questions will be highlighted in this section. Several reports related to general aspects of calculus instruction will first be presented and will then be followed with outlines and discussion of research literature related to student understandings of calculus. Interpretations in terms of constructivism will be included to demonstrate the validity of this perspective for studying student learning in calculus.

A primary reason for a need to research undergraduate calculus instruction is that calculus is a critical course for many undergraduate students. It is critical because it "holds a commanding position in the early undergraduate experience of students hoping to go on in science, engineering, business, and other fields" (Cipra, 1988; p.1491). Calculus is a required course for many students, yet as many as 30% to 50% of them drop-out or fail (Peterson, 1987; Cipra, 1988). In addition, Peterson and Cipra reported that students that do pass calculus tend to perform poorly with respect to calculus skills and the use of calculus ideas.

Reasons cited for this lack of success of calculus instruction include: (Cipra, 1988) (1) many students have a weak background in algebra, geometry and trigonometry (knowledge areas prerequisite to calculus), (2) standard textbooks emphasize rote and repetition in learning, (3) large class sizes are not conducive to effective instruction in that they allow only minimal interaction between students and an instructor, and (4) many instructors have little or no training in the teaching of mathematics, or are uninterested in teaching calculus. Woods (1929) noted similar problems with calculus instruction, including weak student backgrounds, lack of teacher preparation, an emphasis in calculus on computation rather than content and purposes, and students' general lack of interest or motivation to devote time to studying. Thus, problems associated with calculus instruction are not a phenomenon of the last decade only. The fact that similar problems to today were perceived over a half century ago highlights a need for research into undergraduate calculus instruction.

Unfortunately, although there is some agreement on what problems exist in calculus instruction, there are numerous disagreements and difficulties associated with solutions to the problems. Changes that have been suggested include: (see Douglas, 1986; Peterson, 1986) (1) shift the focus of calculus to the fundamental ideas of calculus, rather than emphasizing drill in routine skills and technique, (2) integrate applications into the body of calculus courses by reinforcing the role of approximations and problem situations with contexts relevant beyond the field of mathematics, and (3) produce textbooks to support curriculum changes.

A major reason for requiring so many students to study calculus is for its use in other disciplines. Since calculus applications in other disciplines are often nonroutine and a perceived problem with many calculus courses is their emphasis on solving routine problems, it is important to assess if calculus students are able to apply calculus ideas and skills. In response to this issue Seldon et al. (1989) investigated whether students who had attained a C in an introductory calculus course were able to solve problems for which they had not been taught a solution. They gave five nonroutine problems to each of 17 students who had attained a C grade in introductory calculus. The problems required students to use a combination of techniques and concepts taught in introductory calculus, including differentiation rules, tangent lines, roots, limits, and the concept of differentiability. Seldon et al. found that none of the students solved an entire problem correctly, and many of the solution attempts did not make any use of calculus.

These findings imply calculus instruction for these students did not meet the goal of preparing students to apply calculus. It is also noteworthy that the students in this study were not constrained by large class sizes, inexperienced teachers, poor backgrounds, improper placement tests, or other difficulties often associated with calculus instruction. Thus, Seldon et al. concluded the absence of such handicaps does not automatically lead to improvement of calculus students' problem solving skills. This conclusion implies students' problem solving skills are related to other factors.

Since studies similar to that of Seldon et al. have not been done it is clear that more research is needed into how instruction can better facilitate student learning in calculus. In particular, studies of student perceptions of the nature and goals of calculus instruction are needed. Students themselves might see the learning of routine skills and procedures as a prime goal of calculus. If so, attempts to teach broader knowledge and problem solving skills are likely to meet with difficulties because students do not perceive them to be important.

Hirsch et al. (1983) examined homework as an aspect of instruction that might impact on student achievement. They investigated the effectiveness on calculus achievement of two methods for assigning homework. Two classes of a first semester calculus course were involved. One class (N = 24) was assigned homework "distributively" so that review of past topics was incorporated into daily homework assignments. The other class (N = 28) followed a "conventional" model of homework assignment in which daily exercise assignments were related exclusively to the topic taught that day. By the end of each unit the students had all been assigned identical sets of homework questions. To control other instructional variables as much as possible the classes were taught by the same instructor and both were morning classes.

Regression analysis results from three out of four unit tests with pretest results as a predictor showed significant interaction between homework assignment type and calculus achievement. In these three cases the regression line for the conventional homework group was steeper than the line for the distributive homework group. This indicated a distributive homework assignment schedule benefitted students of below average or average pre-calculus background.

Although the results of this single study do not automatically generalize to all introductory calculus students, in conjunction with similar or related research they might be significant. One such study is that of Edge and Friedberg (1984). Edge and Friedberg examined several student background variables and their effects on achievement in introductory calculus. The background (independent) variables were: American College Test (ACT) scores, high school rank, high school GPA, high school algebra grades, scores from an algebra pre-test, sex, birth order, family size, and high school size. A student's grade in a first semester calculus course served as the dependent variable. For replication purposes the analysis was done on three groups of students of sizes N = 235, 157, and 397.

Edge and Friedberg reported that regression procedures indicated the algebra pre-test and high school rank variables were the best predictors of calculus achievement. They concluded that

> . . . the combination of algebraic skills, as represented by the score on the algebra pre-test, and the long-term perseverance and competitiveness, as measured by high school rank, play a significant role in the prediction of achievement in the first semester of calculus (p.136).

Caution must be exercised in the interpretation of Edge and Friedberg's conclusions. It would not be valid to conclude that achievement in calculus is determined solely by algebraic skills and high school rank. Edge and Friedberg did not give details of the nature of grading for the calculus classes used in the research. The findings of Seldon et al. (1989) already discussed indicate that final grades might be highly dependent on the extent that routine techniques or nonroutine problem solving are assessed on assignments and tests. Further, the interaction of homework distribution and final achievement as reported by Hirsch et al. (1983) indicates that students of low to average pre-calculus background can benefit from a distributive model of homework assignment. That is, there appear to be aspects of calculus instruction itself which can impact on student background variables and thereby improve calculus achievement.

These points show a need for research into calculus instruction and its impact on students. In particular, how instructional structure and emphasis both within the classroom and in use of textbook materials can influence student learning. Not only must achievement of skills and routine procedures be studied, the learning of calculus concepts needs to be assessed. This is because understandings of concepts are important for developing abilities to transfer and apply mathematical knowledge (Skemp, 1987).

However, research into student understandings of calculus concepts has been limited. One of the most extensive studies done is that of Orton (1983a, 1983b) in which clinical interviews were conducted with 110 calculus students aged 16 to 22. The students represented a range of achievement levels in mathematics and were representative of all stages of education in Britain in the 16 to 22 age range. The study investigated students' understandings of limits, differentiation, and integration. Two separate one hour interviews were conducted with each student, during which time each student was given a total of 38 calculus tasks. The tasks were presented on written cards and discussed orally.

Interview responses were graded on a five point scale to enable statistical analysis, and were also categorized in terms of student errors and misconceptions. The statistical analysis revealed that students in the age group 16 to 18 experienced similar successes and failures to students 18 to 22 years of age. Students had the most success with items that dealt with standard applications, and the least success with items that dealt with understanding ideas. These findings imply some students possessed knowledge of calculus skills without knowledge of the underlying principles of those skills. Findings from a categorization of errors and misconceptions also supported this conclusion.

Orton followed a categorization scheme attributed to Donaldson (1963). This categorization classifies errors and misconceptions as structural, arbitrary, or executive. First, structural errors are based on a lack of understanding or misunderstanding of the structure of a concept and its connections to related concepts. For example, Orton observed a inability to interpret negative or zero rates of change, and failure to distinguish between instantaneous and average rates of change. He interpreted these findings as a reflection of incomplete knowledge of the derivative concept. Second, executive errors centre around failing to carry out symbolic manipulations although the underlying principles may be understood. An example that Orton observed was differentiation of $y = \frac{2}{x^2}$ to obtain $\frac{-4}{x}$ rather than $\frac{-4}{x^3}$. Finally, arbitrary errors are responses that are either given arbitrarily or fail to take into account constraints given in a question.

This categorization scheme (structural, executive and arbitrary errors) had been developed for use with younger mathematics students, but proved useful to Orton in revealing aspects of students' calculus understandings. For example, Orton found frequent occurrence of structural and executive errors. These included: failure to distinguish between average and instantaneous rate of change, believing a rotating secant disappears to a point rather than a tangent line, treating $\infty$ as an algebraic symbol that can be manipulated in the same way as numerals, and failing to recognize the connection between the exact area under a curve and the use of the limit of areas of approximating rectangles. Orton concluded students had a number of fundamental difficulties with limit, differentiation, and integration concepts. These findings can be said to be supportive of constructivist notions in that they imply calculus instruction is interpreted by calculus students in a variety of ways, and the related meanings are often different than those the instructor intended.

Orton pointed out several additional implications for the teaching and learning of calculus. These were: (1) laying a stronger foundation for calculus by exploring limit concepts earlier and across several years of mathematics instruction, (2) use of calculators to numerically introduce and develop concepts such as slopes of secant lines approaching the slope of a tangent and sums of areas of rectangles approaching the area under a curve, (3) more emphasis on graphical interpretations, (4) inclusion of more of the foundations of calculus so that calculus ideas are introduced informally and then followed by more formal study, and (5) increased awareness by teachers of the importance of stressing concept development while also anticipating and correcting algebraic difficulties. All of these aspects of calculus instruction are worthy of research studies.

A more recent study by Heid (1988) of student learning in calculus partially addressed these issues. This study involved 39 college students enrolled in an experimental calculus course that stressed concepts over technical skills. The course used graphing and symbol manipulation computer software to emphasize concepts, applications, and problem solving. This approach allowed instruction to include a large variety of concept representations, and it encouraged students to discuss and analyze calculus ideas.

Heid collected data from audio tapes of student interviews, observer field notes, questionnaire responses, and copies of student assignments, quizzes, examinations and class notes. She found students in the experimental course showed better understanding of calculus concepts than 100 students in the control group. Specifically, they were better able to express ideas in their own words, and their conceptualizations were broader, clearer, more flexible, and more detailed than those of students in a control group. They also frequently reconstructed facts from basic principles, a feature students in the control group did not display. In addition, the data reported in the article indicated that between the experimental and control students there were no significant mean differences on a final exam of routine skills. Heid interpreted these results as evidence that students can adequately understand calculus concepts without prior mastery of basic calculus skills.

Heid's findings indicate that the nature and role of the conceptualizations students build as a result of instruction can be influenced by the nature of instruction. For example, the learning environment of the experimental group emphasized discussing ideas. This approach required that students attempt to make sense of calculus related language. It also put emphasis on students seeing themselves as a source of truth or validity. Since students in the experimental group built conceptualizations different than those of the control group, ***language use*** and *sources of conviction* appear to be an important factor of the nature of what is learned. Thus, Heid's study shows constructivism can be useful in interpretation of student learning in calculus.

Heid's findings also support Orton's ideas that students' understandings of calculus concepts might be facilitated by use of numerical procedures, graphical interpretations, and informal followed by more formal instruction. The work of Tall (see Tall, 1989) further supports an emphasis on developing calculus conceptualizations through graphical manipulation and interpretation. Tall reported on previous research of students learning calculus with the assistance of computer software. A key aspect of the design of these programs was that they allowed students to construct and manipulate a large variety of graphs. This feature allowed students to develop what Tall referred to as "cognitive roots" or "anchoring concepts". For example, a program designed to allow the user to magnify any part of a graph can help develop the cognitive root of local straightness as follows:

> Tiny parts of certain graphs under high magnification eventually look virtually straight and this provides as (sic) anchoring concept for the notion of differentiability. Non-examples in the program are furnished by graphs which have corners, or are so wrinkled that they never look straight, providing anchoring concepts for non-differentiability (p.39).

The work of both Heid and Tall points to the impact concrete visual images can have on student learning. The anchoring concepts Tall referred to are reminiscent of Johnson's image schemata (Johnson, 1987) in that perceptual interactions within one's environment are seen to provide meaning structures. Since Johnson's ideas deal with the origin of linguistic meaning, the potential usefulness of ***language use***, and hence constructivism, in understanding student learning is once more demonstrated.

Some other studies that have focused on calculus students' conceptual understandings are those of Tall and Vinner (1981), Davis and Vinner (1986) and Williams (1991). These studies differ from those already discussed in that they investigated conceptualizations of the limit, but not of the derivative or integral. Although these studies are not inclusive of all the research literature on student understandings of limit, they do portray congruent findings (for example, see Sierpinska, 1987).

Tall and Vinner (1981) conducted two investigations with students obtaining an A or B in A-level mathematics in Britain. A questionnaire that asked students to explain and define limits was given to 71 students. Another questionnaire on continuity was given to a group of 41 students. Students responded in writing. Tall and Vinner found that students' written responses to questions displayed both intuitive and formal ideas related to these concepts. For example, they found that intuitive ideas about concepts were generally explained in reference to dynamic, sensory-motor processes. The references included phrases such as: "as $x$ gets nearer and nearer to $a$", "as $x$ tends towards $a$", and "there is a jump at the origin". More formal ideas were generally expressed in terms of mathematical symbols, and included statements such as the following:

$$| f(n) - f(n+1) | < \varepsilon \quad \text{for all} \quad n > N_0$$

Further, students' intuitive and more formal ideas tended to be incomplete or inaccurate.

In a later study Davis and Vinner (1986) found similar results. They studied the curriculum and calculus students of a high school program that culminated in a 2-year calculus course. This program emphasized the understanding of ideas before specific related skills, techniques, and definitions were developed. The investigation focused upon the presence and relative strength of students' informal and formal ideas about limits. An unannounced written test asking for an informal description and a precise definition of limit was given to a class of students at the beginning of their second year of calculus. Responses were then analyzed for correct and incorrect ideas.

Davis and Vinner found several misconceptions present in some student responses. These included: assumptions that a sequence must be monotonic, must have a "last" term, cannot reach its limit, and must exhibit an obvious, consistent pattern. In addition, some responses indicated confusion with the following: limits versus bounds, $f(x_0)$ versus $\lim_{x \to x_0} f(x)$, and the important temporal order of $\varepsilon$ and $N$ in the formal definition of a limit. Davis and Vinner interpreted these findings in terms of possible sources of misconceptions. These sources included: (1) the influence of language, particularly when words such as limit, bound, and variable have associations outside of mathematics, (2) an inappropriate assemblage of mental representations of mathematics from previous intuitive, pre-mathematical fragments, and (3) incomplete mental representations for concepts. All three of these interpretations reflect constructivist notions of learning, though Davis and Vinner did not state them in relation to this perspective.

Davis and Vinner concluded that, in spite of an instructional emphasis upon understanding, the students displayed the same misconceptions as those revealed in other studies of beginning calculus students (Tall & Vinner, 1981; Orton, 1983a, 1983b). They also concluded that, since the students had not been thinking about mathematics for over two months, they were more likely to retrieve from memory naive or imprecise notions. This would result in responses that displayed misconceptions. However, Davis and Vinner did not ask questions to attempt to determine if the students had understandings of limits beyond what was revealed in their initial responses.

More recently, Williams (1991) examined 10 college students' understandings of the limit concept and the factors affecting change in those understandings. The students were selected from a group of 341 students from two second semester calculus classes. The entire group was given a short one page questionnaire to ascertain their beliefs about limits. The 10 interview students were chosen on the basis that their questionnaires clearly and unambiguously supported a particular informal viewpoint of limit. The 10 students each met with the researcher for five sessions varying in length from 30 to 60 minutes. During the sessions the students were presented with problems designed to make them confront anomalous limit situations and alternative models of limit. This design was to encourage them to change their models of limit so as to exhibit more accurate and formal conceptualizations.

Williams found students held a procedural, dynamic view of limit that saw limits in terms of evaluation of a function at points successively closer to a particular point. In addition, he found their models extremely resistant to change. Apparent reasons for resistance were that "students often considered the ease and practicality of a model of limit more important than mathematical formality" (p.233). In particular, they used models of limit that allowed them to deal with "the realities of limits in the classroom" (p.233). In other words students saw as sufficient models of limit that allowed them to deal with the kinds of limits they encountered on tests. From a constructivist perspective these findings reveal that the condition of wanting to do well on tests had significant impact on what conceptualizations students built. A consequence of this condition was that several students' "procedural knowledge (eg. substituting values into continuous functions, factoring and cancelling, using conjugates, employing l'Hopital's rule)" (p.233) was largely separated from their conceptual knowledge.

Other factors Williams found that appeared to support resistance to conceptual change were: (1) students viewed mathematical truth as dependent upon the situation, (2) students saw mathematical truth as open to exception and a variety of different and sometimes conflicting simultaneous statements, (3) students valued a model that was simple and practical rather than mathematically precise, and (4) students' had faith in the appearance of graphs once only a few points had been plotted. Williams concluded that changing students' attitudes about mathematical knowledge is a complex task in need of more research. Also, from factors (1) and (2) specifically, one can conclude that calculus students' *sources of conviction* are likely to depend upon their personal interpretations of situations.

In summary, research into the teaching and learning of calculus has not been extensive. Of the studies that have been done, few have focused on students' conceptual understandings. Additionally, although constructivism has not been a theoretical basis of the research done, it appears to be a valid and useful perspective from which to study and understand students' calculus conceptualizations. It also appears that *language use* and *sources of conviction* are valid and useful in interpreting student learning in calculus.

## F. Background to the Research Methodology

Background to the research methodology of the study is presented in this section. The nature and purpose of qualitative research methodology are discussed, along with related reliability and validity issues. The discussions are largely within a general research context. However, some specifics of the methodology used in this study are included. Further details are given at appropriate places in Chapters 3 and 4.

### Selected Qualitative Research Issues

In the last 20 years educational research has been changing so that it is no longer dominated by quantitative methodologies. The shift has been towards increased emphasis on "inductive analysis description, and the study of people's perceptions" (Bogdan & Biklen, 1982; p.xiii). According to Bogdan and Biklen, a reason for this shift was that many educational researchers felt quantitative research had not proven itself adequate in solving educational problems. These researchers therefore began to adapt to educational research a variety of the qualitative research methods that had already developed rich traditions in other fields. These methods included observation and interview methods derived from case study, ethnographic, phenomenological and critical social theory research. Corresponding to this shift, notions characteristic of quantitative research, including measurement, operationalized definitions, and variables, were extended to use in qualitative educational research.

Bogdan and Biklen (1982) outlined several main characteristics of qualitative research, all of which are incorporated into this study to some extent. First, qualitative research is concerned with context, and uses natural settings as direct sources of data. This feature is in contrast to experimental studies in which "variables are manipulated and their effects upon other variables observed" (Campbell & Stanley, 1966; p.1). Second, qualitative research is descriptive, making use of words rather than numbers. Written results therefore use quotations from the data for illustration and substantiation of findings. Third, instead of focusing solely on outcomes or products, qualitative research concerns itself with process. Finally, qualitative research tends to analyze data inductively as opposed to deductively. More explicitly, qualitative researchers build hypotheses and abstractions from the data gathered, instead of collecting data to accept or reject hypotheses generated before a study begins.

Powney and Watts (1987) mentioned a number of features of qualitative research approaches that correspond with those discussed by Bogdan and Biklen. According to Powney and Watts, recent educational research emphasizes the following:

> . . . analyzing an individual's reactions within the normal context in which they might occur. This avoids reducing the complex responses or behaviour of an individual to a single number in a maze of statistical computations (p.21).

Further, Powney and Watts noted that research in education has been adopting an inductive reasoning approach in which propositions and hypotheses emerge from research data.

This method of constructing hypotheses leads to theory that emerges from the specific data collected. Theory generated in this way is referred to by Glaser and Strauss (1967) as grounded theory. Grounded theory is theory discovered from data. Through the use of original categories and relationships arising from data, hypotheses and concepts are derived in the process of doing research. According to Glaser and Strauss, grounded theory fulfills qualities desired of theory because its mode of generation guarantees it is useful for prediction, explanation, interpretation and application. Also, it serves to guide subsequent research and contributes to theoretical advancement. Furthermore, grounded theory is useful for engendering an active role between theory and research. In comparison, research that merely tests or verifies hypotheses assigns a more passive role to theory (Merton, 1968).

Burgess (1982) described the relationship between theory and research similarly to Glaser and Strauss. He viewed the relationship as one of ongoing interaction in which theory is "involved in constant interplay with the selection of research problems, methods of investigation, and with data collection and data analysis" (p.209). Theory can therefore be used in research in the following ways: (1) to provide an idea for investigation and a means to focus a study, (2) to assist in consideration of alternative perspectives and interpretations, and (3) to give guidance for formulation, reconstruction, or identification of new dimensions of research questions. Burgess also outlined two major logical relationships between theory and research. These are hypothetico-deduction and analytic induction. Hypothetico-deductive methods use research results to verify or falsify a theory. In analytic induction generalizations are derived by refinement and abstraction.

This research study uses both these forms of a relationship between theory and research. It is hypothetico-deductive in that results are used as a test of the usefulness of constructivism for studying mathematics learning. However, this study does more than aid verification or falsification of constructivism. It serves to refine and develop constructivist notions. Through analytic induction, concepts arising from constructivist notions are defined, clarified and refined (for example, notions related to *language use* and *sources of conviction*). Associated analysis methods are also developed. Thus, constructivist theory plays a central role in all aspects of this study. It "influences the problem posed, the methods used, the data collected, the analysis made, and the final report" (Burgess, 1982; p.211). In all these areas analytic induction figures prominently.

Inductive analysis is composed of a variety of analytic procedures (Goetz & LeCompte, 1984). Goetz and LeCompte (1984) outlined a number of these procedures in the context of ethnographic and qualitative research designs in education. The procedures they discussed are: analytic induction, constant comparison, typological analysis, enumeration, and standardized observational protocols. All these procedures are used to a certain extent in this study, and frequently used in conjunction with each other. Each procedure is described here, and additional details are incorporated into relevant places in subsequent chapters.

Analytic induction, as already mentioned, centres on developing generalizations from categories and patterns found in research data. The process begins with working typologies and hypotheses developed from initial investigations of data. These typologies and hypotheses are modified and refined through subsequent data examination (Goetz & LeCompte, 1984). In this study analytic induction is used in some way with all forms of the research data, including classroom observation, textbook and exercise assignment, instructor and student interview analyses.

Constant comparison (a procedure credited to Glaser & Strauss, 1967) is a supplement to analytic induction in that it compares emerging generalizations across all research events (Glaser & Strauss, 1967; Goetz & LeCompte, 1984). That is, it involves all modes of the data collected. In this study the various forms of data arise from questionnaires, classroom observations, textbook and exercise assignment analyses, instructor interviews and student interviews. Some partial comparisons are drawn between these contexts in this study's conclusions.

Typological analysis is a process of division of a phenomenon into groups or categories. It can be used for both descriptive and generative purposes (Goetz & LeCompte, 1984). In this study it is used in conjunction with enumerative methods in the data collection and analysis procedures associated with the classroom observations, textbook and assignment exercise documents, and to a certain extent the student interviews. Enumeration procedures count occurrences of specified phenomena. The phenomena or categories of phenomena must be precisely defined so that "what is countable is clearly designated" (Goetz & LeCompte, 1984; p.186). Enumeration can serve descriptive purposes, and can also supplement analytic procedures aimed at generating, refining or verifying hypotheses. Further, once categories and hypotheses have been developed for a study, enumeration can provide evidence to support the existence and validity of categories.

Standardized observational protocols combine data collection and analysis techniques. The strategies require that (Goetz & LeCompte, 1984)

> . . . initial constructive stages of fieldwork be used to develop enumerative instruments in which units of analysis are precisely specified. Phenomena are then coded during observation into previously designated categories of behavior (p.188).

Croll (1986) referred to this method as systemic classroom observation. According to Croll, "the purpose of systemic classroom observation is to provide an accurate description of selected features of activities and interactions in classrooms" (p.9). To outline fundamental aspects of systemic observation as a research procedure Croll stated the following:

> (i) It is explicit in its purpose or purposes and the purposes have to be worked out before data collection is conducted.
> (ii) It is explicit and rigorous in its definition of categories and in its criteria for classifying phenomena into these categories.
> (iii) It produces data which can be presented in quantitative form and which can be summarized and related to other data using statistical techniques.
> (iv)Once the procedures for recording and criteria for using categories have been arrived at the role of the observer is essentially one of following instructions to the letter and any observer should record a particular event in an identical fashion to any other (p.5).

To develop systemic observation techniques a researcher first spends time in classrooms "to get a feel for the aspects that he or she wishes to investigate" (Croll, 1986; p.173). Appropriate observational variables and their definitions can then be defined and incorporated into later classroom observations. Systemic classroom observation techniques were developed in this study from pilot study observations and classroom observations conducted in the first three weeks of the main study.

Since a major feature of systemic classroom observations is that results are reported in quantitative terms, systemic classroom observations can be said to combine qualitative and quantitative research methods. Used in conjunction in this way and similar ways, quantitative and qualitative research methodologies can be used to supplement and mutually verify each other (Glaser & Strauss, 1967). In this study quantitative and qualitative research methods are used in conjunction in the classroom observation and textbook and exercise analysis methods, and in the student interview analyses. Strengths of each approach are therefore incorporated into the study. The precision, clarity and conciseness of quantitative methods are combined with the more holistic, flexible, interpretive features of qualitative methods. The research conclusions are therefore constrained by the quantitative findings, but are not determined by them. This feature avoids the invalidity of a belief that measures of quantity derived from educational situations "are the same thing as measures of importance" (Croll, 1986; p.184).

Since reliability and validity issues are of central importance in both quantitative and qualitative research, they will now be discussed.

## Reliability and Validity in Qualitative Research

In the context of quantitative research the concepts of reliability and validity have been precisely defined and thoroughly discussed (for example, see Campbell & Stanley, 1966; Payne & McMorris, 1967; Thorndike, 1971; Thorndike & Hagen, 1977; Ebel & Frisbie, 1986). Different techniques than those of quantitative research are used in qualitative research to assess the threats to credibility that reliability and validity issues raise. Since this study is primarily qualitative in nature, with quantitative measures arising from qualitative methodology, the discussion in this section focuses on reliability and validity in qualitative research contexts.

Reliability in quantitative research refers to the extent to which research can be replicated. More explicitly, it is the extent of consistency between measurements applied to people in a situation at a point in time and measurements repeated at a different point in time or in a similar situation (Cronbach & Gleser, 1965). Two forms of reliability found in quantitative research are also addressed in qualitative research. The two forms are external and internal reliability.

Internal reliability focuses on the issue of whether, within a single study, multiple observers would agree. More specifically, internal reliability refers to whether multiple observers would agree in the ways they match the research data to previously generated and defined constructs (Goetz & LeCompte, 1984). Three of the five common strategies outlined by Goetz and LeCompte for use in reducing threats to internal reliability are a feature of this study. These features are: low-inference descriptors, peer examination and mechanically recorded data. Two other strategies recommended by Goetz and LeCompte, multiple researchers and participant research assistants were not used, the first for financial reasons and the second because it was not appropriate to the study.

Low-inference descriptors such as verbatim accounts of interviews, concretely and precisely phrased descriptions from observation notes, direct quotations from documents, and other raw data contribute to a study's reliability. They also contribute to its validity. When used in research reports they facilitate a reader's ability to accept, reject, or modify a researcher's analysis and conclusions. Croll (1986) discussed the value of using low-inference variables in systemic classroom observations. According to Croll, low-inference variables are categories into which observations can be unambiguously coded. These categories are clearly defined so that criteria for classification into categories do not rely on an observer's affective, evaluative or other personal responses. Since observer judgement is therefore reduced to a minimum, reliability is increased.

Goetz and LeCompte (1984) noted that mechanically recorded data enhance reliability by preserving data in its nonabstracted form. Any coding or classifying that follows can be checked for consistency, and constructs generated can easily be revised. In addition, peer examination that arises from making results public enhances reliability in that it encourages researchers to "provide sufficient information in their reports for them to be reviewed adequately" (p.220).

External reliability in qualitative research "addresses the issue of whether independent researchers would discover the same phenomena or generate the same constructs in the same or similar ways" (Goetz & LeCompte, 1984; p.210). Goetz and LeCompte outlined five major factors which qualitative researchers should address to enhance the external reliability of their data. All of these factors are addressed in this research report. The factors are: researcher status position, informant choices, social situations and conditions, analytic constructs and premises, and methods of data collection and analysis.

A researcher must report his or her role and status position within the research setting investigated because other researchers will be inhibited in drawing comparisons to their own studies unless they hold similar positions within the research environment (Goetz & LeCompte, 1984). This point is particularly important in studies where access to information and the type of information gathered is highly influenced by the social relationships a researcher has with individuals in the research study. For example, in this study student interviews conducted by an instructor for a course would likely have yielded different responses than the interviews conducted by the researcher. The reason is that, unlike an instructor, the researcher's relationship with students did not involve the determination of course grades. Similarly, a qualitative researcher must also explicitly describe the social situations and conditions of research settings. Some reasons for this point, according to Goetz and LeCompte, are that under different contexts individuals will see different things as appropriate to reveal, and what individuals say and do might depend on who is present at the time or who the recipients of the information are perceived to be. Which individuals serve as informants or suppliers of information, and the decision process for their choice is also important to establishing external reliability. That is, the individuals who provide data must be identified. Any biases in the subsequent data can then be identified and handled appropriately.

Of particular importance to enhancing the external reliability of this study is delineation of analytic constructs and premises. Outlining theoretical premises and defining constructs developed in the research facilitates replication. Goetz and LeCompte noted that "definitions for concepts should be clear and sufficiently lacking in idiosyncrasy so as to be to be intelligible to other researchers . . ." (p.216). In addition, the researcher should report "which concepts and definitions remained constant throughout the research process and which were generated, developed or refined through data collection, analysis, and interpretation" (p.216). Fulfillment of these guidelines aids comparability between studies. Hence, these guidelines have been followed in Chapters 3 and 4 of this research report.

Also of prime importance to the external reliability of this study is sufficient reporting of the methods of data collection and analysis. The reason is that reliability depends upon whether other researchers would be able to reconstruct data collection and analysis methods. Goetz and LeCompte stated the following in relation to data collection:

> . . . descriptions should specify how observations were recorded, mechanically or by fieldnotes, how field notes were composed, in situ or post hoc, the circumstances under which interviews were conducted, and how material from various sources were integrated into the study (p.217).

These guidelines have been followed in the research design presented in Chapter 3. As well, reliability has been addressed by identification of the analytic procedures. The importance of this aspect was explained by Goetz and LeCompte as follows:

> Simply asserting that analysis has been carefully done is insufficient for establishing the credibility, reliability, and validity of ethnographic efforts. The researcher must clearly identify and fully discuss data analysis processes and provide retrospective accounts of how data were examined and synthesized (p.217).

Validity in quantitative research is concerned with accuracy. It addresses the questions of whether the research measures "what we want it to measure, all of what we want it to measure, and nothing but what we want it to measure" (Thorndike & Hagen, 1977; p.56). Internal validity refers to correct attribution of causality, while external validity is concerned with generalizability. Both of these concepts have been extended to qualitative research contexts.

Internal validity is concerned with the extent to which "observations and measurements are authentic representations of some reality" (Goetz & LeCompte, 1984;

p.210). That is, are researchers observing and measuring what they think they are observing and measuring? Threats to internal validity that must be addressed in both quantitative and qualitative research include: history and maturation, observer effects, selection and regression, mortality, and spurious conclusions (Campbell & Stanley, 1966; Cook & Campbell, 1979; Goetz & LeCompte, 1984). The following explanations of each of these factors are based on presentations found in Goetz and LeCompte (1984).

History refers to change in the overall social setting of a study, while maturation involves the development of individuals (Goetz & LeCompte, 1984). The first of these factors is not a difficulty in this study because the research settings remained relatively constant over the course of the research. The second factor, maturation, does occur in this study. Students necessarily developed academically, socially, and emotionally throughout their term in introductory calculus. However, since this study deals with students' conceptual understandings at a particular point in their development, internal validity is not threatened.

Observer effects come into play as a threat to validity through the role a researcher assumes in a research setting. A researcher's position affects the data in that research participants might act differently than normal while in the presence of the researcher. To reduce such artificial responses it is necessary for the researcher to establish "sufficient residence in the field" (Goetz & LeCompte; p.224). Residence in the field was achieved in this study by the researcher's attendance of classes for at least two weeks before the related, coded classroom observations were made use of in the final analyses. In addition, since instructor interviews were done in October and student interviews were done at the end of the term, the researcher was not a stranger to the interviewees at the time of the interviews.

Selection effects were dealt with in this study in that student interview volunteers were sought so that they represented a range calculus achievement levels. The selection also included a balance of males and females, and students representative of a range of mathematics backgrounds and present academic programs. The research settings were selected to provide a variety of instructional settings and approaches to instruction. Mortality was a difficulty in this study only in that no interviews were done with students who dropped out of calculus before the end of the term. However, some of the interviews were conducted with students who might have dropped out, but had decided otherwise in the hopes they might pass the course.

Spurious conclusions occur when it is concluded that associations among phenomena exist when in fact they do not, or vice versa (Goetz & LeCompte, 1984). Difficulties associated with spurious conclusions are dealt with in this study in more than one way. Data is examined for a variety of plausible causes of observed phenomena, evidence supporting all conclusions is explicitly reported, and the evidence supplied makes extensive use of primary data from classroom observations, textbook and exercise assignments, instructor interviews and student interviews.

The other main type of validity, external validity, refers to the degree to which a researcher's representations of some reality "can be compared legitimately across groups" (Goetz & LeCompte; p.210). That is, "to what extent are the abstract constructs and postulates generated, refined, or tested" (p.221) by researchers applicable across groups? Threats to a study's external validity are effects that block or reduce a study's comparability and translatability. Comparability refers to the extent to which a study adequately defines and describes its components, including research setting characteristics, analysis units, and the generation of concepts. Investigations of related issues cannot be compared to a particular study unless that study supplies sufficient explication of its components. Translatability refers to how accessible and understandable to other researchers are the definitions, research techniques, and underlying theories of a study. Thus, "external validity depends on the identification and description of those characteristics of phenomenon salient for comparison with other, similar types" (p.229). Efforts are made throughout this report to fulfill all these external validity specifications.

### G. Summary

Constructivism has been emerging as a prominent theoretical research basis in mathematics education. It is a theory of knowledge that sees mathematics learning as an active, constructive process in which an individual builds knowledge for himself or herself. According to constructivism, an individual constructs internal schemata from interaction with her or his physical and mental environment. In this way constructivism views learning as an adaptive process in which an individual constructs a viable model of the world.

Constructivism sees learning as an individual, constructive process, but constructivist literature does not discuss the variety of ways an individual might go about this construction. In particular, constructivist literature does not adequately disuss how the nature of an individual's schemata are likely to differ according to differences in personal and situational conditions. Although constructivism highlights interactions with one's environment and the building of viable internal conceptualizations, constructivist literature does so from a single all encompassing perspective. Constructivism speaks of learning as personally meaningful and relevant. However, it is not clear that all individuals necessarily view all their learning as personally understandable or meaningful. For example, an individual who learns particular information by rote memorization is likely to build different conceptualizations of the material than does an individual who learns the same information by actively trying to make sense of it. Both individuals will have "built" some sort of conceptualization, but the nature and role of these conceptualizations are likely quite different. At present, constructivist literature does not discuss the nature of such individual distinctions in learning. Thus, more research and subsequent refinement of constructivist ideas are needed.

More research is also needed in the adequacy and completeness of the use of constructivist theory to describe and give insights into mathematics learning. In particular, radical constructivism might be limited in its use as a perspective from which to view mathematics classrooms and related learning. It is not clear that radical constructivism's rejection of realism gives a valid or practical description of mathematics students' and mathematics teachers' views of mathematics. For example, constructivism sees objective mathematical knowledge as lying in the "shared rules, conventions, understandings, and meanings of the individual members of society, and in their interactions" (Ernest, 1991; p.82). If mathematics students and teachers do not share this perspective of negotiation of knowledge, then it might be that constructivism is an inappropriate theory on which to base mathematics education research. However, more research is needed before this aspect can be determined.

Two other aspects of mathematics learning that arise from constructivist theory that are in need of research are *language use* and *sources of conviction*. *Language use* is of importance because a constructivist perspective sees mathematics as grounded in "linguistic knowledge, conventions and rules (Ernest, 1991; p.42). The literature on the relationship of language to learning highlights the role of metaphor (Johnson, 1987; Pimm, 1987). In particular, natural everyday language is important to mathematics learning from a constructivist perspective because constructivism sees previous experience, and hence language experience, as a key component of an individual's construction of conceptualizations (Halliday, 1985; Pimm, 1987). Language is therefore a valuable component of the building of meaning from experiences. Further, it is largely through language that an individual constructs mathematics conceptualizations.

*Sources of conviction* are important to mathematics learning viewed from a constructivist perspective because constructivism sees mathematics knowledge as dependent upon a social sharing of decisions pertaining to truth and validity. The nature of what an individual learns will therefore be reflected in the ways he or she attributes truth and validity. Research into the nature and role of notions related to *sources of conviction* is needed. Some theoretical discussions of similar concepts have been done, including discussion of intuitive and school knowledge (West & Pines, 1985), private and public understandings (West e al., 1985), and subjective and objective knowledge (Ernest, 1991). However, there has not been research into the ways mathematics students attribute truth or validity, and how these features influence conceptualizations.

There have been a few studies of student achievement in calculus (Hirsch et al., 1983; Edge & Friedberg, 1984; Seldon et al., 1989). As well, although there has been some research into students' conceptual learning of calculus, research into the teaching and learning of calculus has not been extensive. However, previous research into student learning in calculus can be re-interpreted from a constructivist perspective. This fact indicates research from a constructivist perspective would be both valid and insightful for studies of student learning in calculus.

# 3. RESEARCH PROCEDURES

## A. Introduction

In this chapter the design of the study and the research procedures are described. The pilot study and its purpose are also described.

To address the first three research areas related to students' *language use, sources of conviction* and manner of construction of conceptualizations, clinical and personal interviews were done with 5 or 6 students from each of the three post-secondary institutions. The in depth clinical interviews involved students in oral and written responses to a number of calculus problems focusing on calculus skills and concept interpretations. The personal interviews were aimed at investigation of students' mathematics backgrounds, perceptions of calculus, perceptions about the role of language and mathematical notation in their learning, and ways of determining mathematical correctness or validity.

The fourth research area, the impact of different instructional approaches upon students' *language use, sources of conviction* and manner of construction of conceptualizations was addressed through analysis of student interviews, as well as examination of classroom and textbook instructional events at the three institutions. Students' problem responses illustrating various interpretations and solutions were interpreted for possible relationships to the three instructional approaches. The small number of students interviewed at each institution did not permit statistical analysis or definitive answers to the question of the effect of instruction upon students' calculus learning. However, the examination of student's problem responses gives insight into the potential impact of each of the instructional approaches on students' learning.

The fifth, underlying research area, description of the ways the three instructional approaches translate into instructional events was addressed through five research activities. Specifically, to obtain a comprehensive description of the three instructional settings and approaches to instruction the following activities were undertaken:

(1) A Background Questionnaire at the start of the school term was administered to determine class characteristics. Any relationships between class characteristics and the impact of instruction on student learning might therefore be determined.

(2) An End of Term Questionnaire was administered to gain insight into how students' views of calculus and experiences in calculus might be related to the three instructional approaches.

(3) Instructors were interviewed to obtain articulation of the philosophy of each instructional approach, as well as each instructor's interpretation of the instructional approach.

(4) Classroom observations were conducted to determine how instruction was delivered to students.

(5) Analysis of textbooks and exercise assignments was undertaken to determine in what manner the materials reflected the instructional approaches, and to what extent the materials were similar across the three approaches.

### B. Research Setting and Pilot Study

The study was undertaken with three undergraduate introductory calculus classes. One class at each of the following post-secondary institutions was involved: (1) a large urban university (approximately 25,000 full-time students), (2) a small college (about 800 students) in a town (population 13,000) about 100 kilometres from the city where the university is located, and (3) a small urban college (about 2000 students) in the same city as the university. Henceforth, the university will be referred to as Alpha University, and the colleges will be referred to as Beta College and Gamma College, respectively. These are pseudonyms adopted for ease of referral to these institutions.

The introductory calculus course at Alpha University is representative of such courses across North America. A standard textbook is used, Single Variable Calculus (J. Stewart, Brooks/Cole Publishing Company, 1987). In comparison, the introductory calculus courses at the colleges make use of unpublished textbooks written by instructors at each of these institutions. The general content topics in these textbooks are similar to those of the university textbook, but units are not necessarily covered in the same order or by the same approach. However, the courses at the colleges are similar enough to the course at the university to have been granted credits transferrable to the university. More details on these courses, as well as the course at the university are given in Chapter 4.

A pilot study that had the following objectives was conducted: (1) to gain experience observing calculus classes, while developing methods for recording observations, (2) to field-test the Background Questionnaire and a variety of written response problems, (3) to gain experience in conducting student interviews, while field-testing a variety of clinical interview problems, and (4) to refine the research questions and establish their empirical feasibility.

The pilot study was carried out with two introductory calculus classes. One class was at Alpha University and the other was at Gamma College (urban college). The pilot study took place in May and June 1990, during Spring Session courses at these institutions. Due to the intense time scheduling of these courses (2 to 4 hours per day) it was not possible to also conduct pilot work at Beta College at this time. The classes observed during the pilot study were taught by different instructors than the research classes of the main study. Each class was observed 2 to 3 hours per day for a 3 week period. Written response problems were given to all students in the college class on four occasions, and to the university class on one occasion. Each problem set required 20 to 30 minutes for all students to complete. Clinical interviews were done with 3 students at the college. Each student was interviewed on 3 occasions with each interview lasting 45 to 60 minutes. Clinical interviews were done with 4 students at the university. Each was interviewed once and the interviews lasted 60 to 90 minutes.

The pilot study was conducted from a constructivist perspective. That is, the researcher approached interpretation of classroom observations and student interview responses from the perspective that students construct individual understandings of instructional events. The feasibility of studying student learning in calculus from this perspective was therefore tested. Findings from the clinical interviews led the researcher to develop the concept of *sources of conviction*, and to subsequently focus the research questions upon *language use* and *sources of conviction*. This focus was taken because students' *language use* and *sources of conviction* as displayed in the interviews appeared to be revealing of students' conceptualizations and reasoning patterns.

No textbook analyses were done in the pilot study because the textbook used for the Spring Session university class was different from the text to be used during the main study, and changes were to be made to sections of the text for the college textbook. The End of Term Questionnaire was not fieldtested because a need for this questionnaire was not determined until later. The final set of clinical interview problems was determined from findings of the pilot study. Additionally, the decided format of a single interview followed by a short follow-up interview was so as to compromise between the originally intended format of several interviews with each student and the difficulties inherent in having students commit to and be able to schedule interview times. Finally, since the written problems given to all students during the pilot study infringed upon instructional time and did not produce informative results, this portion of the research was dropped from the main study.

## C. Research Instruments and Data Collection

In this section the rationale for and development of the Background and End of Term Questionnaires, classroom observation and textbook and exercise assignment analysis methods, and instructor and student interview methods and questions are described.

### Background and End of Term Questionnaires

The Background Questionnaire can be found in Appendix A. The purpose of this questionnaire was to determine class characteristics, and any relationships between class characteristics and the impact of instruction on students' learning. The questionnaire gathered information on a student's mathematics background and grades, size and location of high school attended, language background, career plans, major field of study, reasons for taking an introductory calculus course, reasons for choosing the post-secondary institution presently attended, and attitudes towards and impressions about mathematics. From this information a background profile of each class as a whole could be formed. The characteristics and degree of similarity of the three classes at the start of the school term could then be ascertained. The questionnaire was created by the researcher and was field-tested during the pilot study for clarity and ease of completion.

During the first week of classes the questionnaire was given to all students in the three classes, except those students absent from class the day the questionnaire was given to their class. During the following week the researcher attempted to have students who had been absent complete a form on their own time to be returned later. Not all these students returned the forms. However, the exact number missing for each class is not known because enrollment in each class varied throughout the term as students withdrew from the course. Additionally, during the first week it was also possible for students to join the class.

Before administration of the questionnaire the researcher explained verbally to the students the purpose and nature of her work. Students were told the researcher was a mathematics teacher doing research towards a doctoral degree. They were told their class was part of a study into student learning in calculus in terms of how certain factors play a role in learning calculus. They were also told that regular classroom instruction would not be disrupted, no special knowledge or skills were needed for participation in the study, and participation in the study would not figure in the determination of grades. It was also explained to the students that the researcher would be attending classes regularly, and during these times would make notes on things occurring during instruction. Students were also informed that the researcher would at a later date ask for interview volunteers, and that details about the interviews would be given at that time. Finally, it was emphasized to students that participation in interviews or completing the questionnaires was voluntary, and that confidentiality would be maintained at all times. These points were further noted to students in the written explanation on the consent form and background questionnaire instructions attached to the front of the Background Questionnaire (Appendix A).

The End of Term Questionnaire can be found in Appendix B. It was designed to gain insight into how students' views of calculus and experiences in calculus might be related to the three instructional approaches. It gathered information on students' study habits, attitudes towards calculus and impressions about their calculus course. Many of the questions were chosen to overlap the personal interview questions asked of the interviewees (see Appendix J). These questions included those that related to apprehension about calculus, perceived usefulness of calculus, ideas of what calculus is about, present achievement level in calculus, time spent studying, and indication of specific study practices. A main reason for the End of Term Questionnaire was to determine any differences between the three classes on the various items. In conjunction with data collected from class observations, textbook analyses and interviews, any significant differences found might then be ascribed to the impact of the different instructional approaches and settings. Another purpose of the End of Term Questionnaire was to ascertain if the related information obtained from the interviewees was representative of their respective classes.

### Classroom Observations

The classroom observations were aimed at providing a description of instruction as delivered to each of the three classes. Classroom observations therefore provided description of the ways the three instructional approaches translate into instructional events. They also were used in addressing the issue of the impact of different approaches to instruction on students' *language use*, *sources of conviction* and manner of construction of conceptualizations. In particular, data was gathered to provide information on: (1) conditions of the instructional setting, (2) relative time spent on concept development and use of examples, (3) ***language use***, and (4) ***sources of conviction***. The classroom observation methods were a combination of systemic classroom observation techniques (Croll, 1986) and qualitative fieldnote procedures (Goetz & LeCompte, 1984) (also see Chapter 2). Systemic observation methods were used to concisely describe and summarize aspects of classroom instruction related to ***language use*** and ***sources of conviction***. Fieldnote procedures were used to capture as complete a picture as possible of what occurred in each class. They also provided numerous examples of specific instructional events in terms of the sequencing of ideas and what was said or written to explain or justify ideas. In this way the fieldnotes enabled description and comparison of instruction in the three classes. Thus, the classroom observation procedures allowed the researcher to describe specific events of instruction, while simultaneously quantifying instructional features related to the research questions.

Classroom observations were done for a 13 week school term from September to December 1990. Each of the three classes had four 50 minute periods scheduled per week. For the class at Alpha University three of these hours were classes with the instructor, while the remaining hour was a lab supervised by graduate students in mathematics. Classroom observations of the regular class hours at Alpha University were done once or twice per week. Lab hours were visited on six occasions. For the courses at the colleges all four weekly meetings were with the instructor. Classes at Beta College were observed once per week, and classes at Gamma College were observed twice per week. The fewer number of visits to Beta College was due to the distance that had to be travelled to reach Beta College. Within a Monday to Friday schedule the general sequence of classroom observations was: Gamma College, Alpha University, Beta College, Gamma College, Alpha University.

The researcher attended classes as an observer, rather than a participant observer (Goetz & LeCompte, 1984). She made lecture notes alongside other students, but did not actually participate as a student in that she did not ask questions or respond to instructor questions and did not complete assignments or exams. The only exception to this was during group problem solving sessions at Gamma College (further details on these problem solving sessions are in Chapter 4). The researcher would join with a group in these sessions and participate by occasionally asking questions or asking for explanations.

The way fieldnotes were made is now described (also see Appendix D). While attending a class the researcher would write down all information the instructor wrote on the board, and would also write down as much as possible of the phrases, sentences or questions spoken by the instructor. The researcher kept these features properly identified by using printing for recording blackboard notes and writing for recording spoken words. Words spoken by students were recorded in writing with a hyphen in front of them. Additional notes and comments were also recorded with a hyphen in front of them. These additional notes included general impressions of some instructional incidents or specific student behaviours. Examples included the following: "teacher asks lots of questions though he generally answers them himself", "teacher talks almost nonstop, constantly explaining what he is doing", "teacher explains by using the metaphor of a light switch", "ideas presented in general form first", "many students write notes right in text, on left side of the page" and "several students remain after class to ask questions".

Following a classroom observation (within one or two hours, and always before attending any other class) the classroom observation notes were summarized on Classroom Observation Summary Sheets (see Appendix C). The Classroom Observation Summary Sheet was developed from a systemic classroom observation perspective (Croll, 1986) to reduce the observation data and focus it on factors related to the study's research questions (*language use* and *sources of conviction*). For each of the three classes a sample of notes taken during an observation and the corresponding summary sheets can be found in Appendix D.

The variables represented by each column of the Classroom Observation Summary Sheet were generated from pilot study work and classroom observations of the first three weeks of the main study. These variables were developed as much as possible as low-inference variables so that observer judgement would be minimal and coding would be unambiguous (Croll, 1986). This feature was also to enhance the reliability of the research methods.

The information recorded for each variable (column) of the Classroom Observation Summary Sheet will now be defined. One or more possible category codes was entered in each column. One entry was made under each of the Time and Event columns, but more than one entry was possible under the Language and Sources of Conviction columns. These are described below, and the codes used for each category are given in brackets after the name of the category. A summary of the descriptions can be found in Appendix E.

### Time

Observations were analyzed in two minute intervals. This structure was decided upon because it allowed segments of instruction to be coded as wholes, rather than isolated pieces of written or spoken events. That is, the two minute time intervals provided opportunity for instructional events to be coded as meaningful, coherent units. Shorter time intervals did not allow this meaningful segmentation, but rather, split instructional events into seemingly unconnected pieces. However, occasionally a two minute time interval was split into two one minute intervals, and coding was done for each one minute interval. This split occurred when instruction changed within a two minute interval between presentation of a concept and presentation of an example. The time at the start of an interval was recorded in the Time column. All classes were approximately 50 minutes in length, yielding about 25 samples of instructional events per classroom observation.

Event

One of three category codes was entered in this column, along with mention of the content of the particular event. The three categories are described below. They emerged from the pilot study as suitable categories for describing events within a lecture format for calculus instruction. Within the first three weeks of classroom observations of the main study they continued to be suitable, and were therefore retained in their original form.

(1) (CP) Concept Presentation: The instructor develops or further explains concepts. This presentation might be in a general form, or might also be in conjunction with a specific example. Presentation of proofs is included in this category.

(2) (EX) Example: The instructor works through an example exercise problem to exemplify an idea, demonstrate a calculation, or solve a multistep problem.

(3) (O) Other: This includes administrative details such as collecting or handing back assignments, determination of test dates, or other events that are not explicitly instructional. Also included here are times when a class begins late or finishes early.

Language (Written or Spoken)

Codes were entered in the Spoken Language column only if the spoken language of instruction differed from the written language. That is, if the spoken language was mostly reading of what was written, then the spoken language was not coded. Two types of categories were recorded under the language columns. One category was Language Type, while the other related to the Context of the material presented.

A. Language Type

(1) (TL) *Technical Language:* The language used is language generally accepted as proper and correct by the mathematics community at large.

(2) (EL) *Everyday Language:* The language used is not generally recognized by the mathematics community for use in unambiguous mathematical discourse. These words and symbols might or might not be mathematical in nature and are often words found in daily English language use.

The two language type categories emerged from the pilot study, and their description continued to be suitable for classroom observations in the main study. Examples from the pilot study of technical language use include: "as x increases without bound", "translate the function along the positive x axis", and "let L be a nonvertical straight line in the plane". Corresponding examples of everyday language are: "as x gets very, very big", "shift the function to the right", and "let's look at a line L that doesn't go straight up and down".

B. Context

(1) (MC) Mathematical Context: The circumstances of the instruction are mathematical in nature and this is made explicit through the language used.

(2) (PC) Physical Context: The circumstances of the instruction refer to or use sensory-motor experiences of the world. Included here are graphs or diagrams, and mention of physical objects such as cars or hills.

(3) (CF) Context Free: The instruction is rule-governed, without reference to the origin of the rules.

The Context category was included under the language column because it is through association with preceding and following language (words and symbols) that the meaning of an instructional incident is constructed. Reliability on the part of the researcher for designation of an event as displaying Physical Context (PC) was necessarily high. This is because reference to or use of sensory-motor experiences was necessarily explicit. In terms of designation of an event as either Mathematical Context (MC) or Context Free (CF), reliability was established by the researcher designating an event as displaying a Mathematical Context (MC) only if the instruction explicitly stated what mathematics concept or concepts were involved. Otherwise, the Context Free (CF) code was entered on the summary sheet.

On a more general level, reliability of the researcher's categorization of classroom observations was established in three ways. First, the sequence of classroom observations (Gamma College, Alpha University, Beta College, Gamma College, Alpha University) was such that the researcher constantly compared observation categorizations with previous ones from a different institution. Second, for the first eight weeks of observations, beside a code entered on a summary sheet, the researcher entered a short reason for the choice of that code. Third, whenever the researcher encountered difficulty in the determination of the appropriate category she examined previous observation notes and corresponding summary sheets for similar circumstances. The decision made was therefore consistent with previous decisions.

At this point it must be noted that according to constructivism it cannot be assumed that students were constructing the same contexts as those the researcher interpreted as being explicit in the instructional event. The researcher recorded if the instruction explicitly displayed circumstances of a particular context. What context individual students inferred is not known.

Convictions (Written or Spoken)

Four categories were used here, dependent upon how the instruction validated its statements and decisions. As with the Language columns, entries were made under the spoken column only when what was spoken was more than a reading of what was written.

(1) (IM) Internal/Mathematics: Truth and validity claims are made in reference to previously established mathematics, or through logical necessity.

(2) (IE) Internal/Experience: Truth and validity claims are made in reference to sensory-motor experiences. These references include use of graphs or diagrams, and reference to physical objects.

(3) (ER) External/Rules: A rule or rules are followed that either have not been previously justified or are not used with justification as to the choice of particular rules.

(4) (EO) External/Other: Truth and validity claims are made without any source being given, or the source acknowledged is the textbook, lab manual, or other document.

As with the context category, it must be noted here that the researcher recorded if the instruction explicitly displayed circumstances from which a particular source of conviction could be claimed. What sense students actually made out of the events is not known. For example, although instruction might have justified particular procedures through reference to previously taught mathematics, an individual student might easily have interpreted the entire occurrence as "rules" to be followed to get "correct" answers. The internal-external designation was used to distinguish between *sources of conviction* which might guide a student towards seeing himself or herself as a source of the determination of truth, versus *sources of conviction* which emphasize rules or external authority.

Other Observations

Additional notes from the classroom observation fieldnotes (those preceded by a hyphen) were recorded in this column if they were indicative of frequently occurring events in that classroom. A primary item entered was a note of when a student either asked or answered a question.

After the school term was completed the data on the Classroom Observation Summary Sheets was used for further instructional analysis. These analyzes and their results will be discussed in Chapter 4.

### Textbook and Exercise Assignment Analysis

For each of the three courses, textbook and exercise assignment analysis involved exercise assignments, and sections of the textbook covered in the course. At Alpha University sections of the lab manual covered in the course were also analyzed. Thus, whenever the "textbooks" are referred to, the Alpha University lab manual is implicitly included in the reference.

The textbook and exercise assignments were examined to further complete the descriptions of the instructional events obtained from the classroom observations. Thus, relative time spent on concept development and use of examples, *language use* and *sources of conviction* were variables focused on in the generation of textbook and exercise assignment analysis typologies.

The textbook and exercise analysis procedures were an adaptation to written documents of systemic classroom observation techniques (Croll, 1986). Thus, the textbook and exercise assignment analyses involved generation of well-defined categorization variables appropriate to textbook content. According to Borg (1963), "content analysis is a research technique for the objective, systemic, and quantitative description of the manifest content of communication" (p.256). It must involve specific and well-defined categories so that "different researchers of comparable skill could use the procedures independently and obtain very similar results" (p.257).

To allow as much uniformity as possible between the textbook, exercise assignment and classroom observation analysis categories, the Textbook Analysis Summary Sheet was designed (see Appendix N). Textbook analysis could not however be divided into two minute time intervals. Instead, the material within a section of the text was divided according to Events (as outlined below), regardless of the written length of the event. The Language and Convictions variables (columns) on the Textbook Analysis Summary Sheet are defined similarly to the same variables on the Classroom Observation Summary Sheet. However, since all material presented in the textbooks was written, the "spoken" columns were eliminated. The Event and Type variables (columns) are explained below. A summary of descriptions of the variables associated with the Textbook Analysis Summary Sheet can be found in Appendix F.

Event

One of three categories was recorded in this column, two of which are the same categories as found in the corresponding column of the Classroom Observation Summary Sheet. The three categories are:

(1) (CP) Concept Presentation: The text material develops or further explains concepts. This might be in a general form, or in conjunction with a specific example. Presentation of proofs is included in this category.

(2) (EX) Example: The text material is an example exercise problem.

(3) (EXC) Exercise: The text material is an exercise for the student to work through on her or his own.

If certain exercises were explicitly assigned to students in the course syllabus or by the instructor, then only those exercises were analyzed. Otherwise, all exercises in that section of the textbook were considered. Exercises assigned to the students that were not in the textbook were also examined. Exercise events were not analyzed for ***language use*** and ***sources of conviction*** because language context and sources of conviction could not be determined independently of a solution to an exercise.

Language

These definitions are direct translations from the corresponding definitions for the Classroom Observation Summary Sheet.

A. Language Type

(1) (TL) *Technical Language* : The language used is language generally accepted as proper and correct by the mathematics community at large.

(2) (EL) *Everyday Language* : The language used is not generally recognized by the mathematics community for use in unambiguous mathematical discourse. These words and symbols might or might not be mathematical in nature and are often words found in daily English language use.

B. Context

(1) (MC) Mathematical Context: The language of the textbook event is explicitly mathematical in nature.

(2) (PC) Physical Context: The textbook event refers to or uses sensory-motor experiences of the world. Included here are graphs or diagrams, or mention of physical objects.

(3) (CF) Context Free: The textbook event states rules, ideas, or procedures without reference to their origin.

## Sources of Conviction

The *sources of conviction* categories are translations from those of the Classroom Observation Summary Sheet. However, the External-Other (EO) category was dropped because it did not apply to the textbook presentations.

(1) (IM) Internal-Mathematics: Truth and validity claims are made in reference to previously established mathematics, or through logical necessity.

(2) (IE) Internal-Experience: Truth and validity claims are made in reference to sensory-motor experiences. This includes use of graphs or diagrams, and reference to physical objects.

(3) (ER) External-Rules: A rule or rules are followed that either have not been previously justified, or are not used with justification as to the choice of the particular rule or rules.

## Type

Entries were made in this column only when either the Example (EX) or the Exercise (EXC) category was entered in the Event column. Analysis of the examples and exercises was approached from a constructivist perspective in terms of examining how the examples and exercises might affect students' construction of conceptualizations. An analysis of mathematical tasks as being either routine or nonroutine was initially chosen as a possible categorization scheme for examples and exercises (Christiansen & Walther, 1986). This scheme places tasks within the following two-column categorization:

| *Routine tasks* (exercises) | *Nonroutine tasks* (problems) |
|---|---|
| Recognition exercises | Process problems |
| Algorithmic exercises | Open search problems |
| Application exercises (word problems) | Problem situations |

Christiansen and Walther define routine tasks as tasks for which a procedure leading to a solution is known. Nonroutine tasks are tasks for which a procedure leading to a solution is not known. A necessary component of a nonroutine task is therefore a degree of uncertainty or undecidedness as to a procedure for its solution. According to Christiansen and Walther, both routine and nonroutine tasks are important for mathematics instruction. Performance of routine tasks is a means of consolidating knowledge and skills, while nonroutine tasks provide for the following:

> - *optimal conditions* for a cognitive development in
> which:
> - new subjective knowledge is constructed by the individual;
> - items of earlier acquired knowledge (information like awareness) are recognized and evaluated by the individual - in new perspectives, with new potentials, in new mutual relationships - and are reorganized and restructured into an enlarged and consolidated body of knowledge (p.275).

Although these notions reflect constructivist views in their emphasis on individual construction of knowledge, the researcher did not initially know if Christiansen and Walther's two-column categorization scheme could be directly applied to the textbook and exercise analyses of this study. However, to attempt to apply this scheme to textbook examples, the following question was asked of each example: "Could students use this example to learn by imitation?" That is, could students duplicate the steps followed in the example to work through a variety of exercise questions similar to the example? Asking these questions proved useful for designating an example as either routine or nonroutine. However, to reflect the nature of the question the researcher asked of each example, it was decided to name the categories "Imitation" and "Non-imitation".

Difficulties arose in further attempts to use Christiansen and Walther's two-column scheme. The subcategories of the two columns were not appropriate for the calculus textbooks examined. Thus, through examination of the examples in these textbooks the researcher developed a new set of subcategories. These will be presented and discussed in Chapter 4.

Attempts to use Christiansen and Walther's scheme for exercise examination also encountered difficulties. Initially, the following question was asked of each exercise: "Could students do this exercise by recall, or by simply following rules or procedures?" The researcher found this question could not be answered for all exercises. Thus, Christiansen and Walther's scheme was not adequate for an analysis of calculus exercises. The adapted scheme that emerged in its place as the researcher continued examination of the exercises is given in Chapter 4. This scheme involves three main categories and a number of subcategories. Some of the subcategories correspond to those of Christiansen and Walther's scheme, while others emerged during the analysis process. Thus, the textbook analyses combined inductive, typological procedures (Goetz & LeCompte, 1984) with systemic techniques (Croll, 1986).

## Instructor Interviews

An interview was conducted with each instructor to obtain articulation of the philosophy of each instructional approach, as well as each instructor's interpretation of the instructional approach. These were respondent interviews in that interview topics were established by the researcher before the interviews took place (Powney & Watts, 1987). The "locus of control" for what happened throughout the interview was therefore the researcher's responsibility. The instructor interviews were done to provide data on instructors' views of factors behind what occurred in each instructional setting. The interviews focused on an instructor's past and present teaching experiences, time made available outside class for interactions with students, beliefs about teaching and learning, perceptions of calculus students' abilities and what motivates them, impressions about the calculus course and textbook, and bases for decision making on instructional emphases and strategies. The specific questions asked in the interviews can be found in Appendix G. These questions served as a guideline for the interviews. Supplementary questions asking for expansion and clarification then arose from an instructor's responses.

The interviews were done during the first three weeks of October. This time period was chosen because it gave the researcher exposure to the three calculus courses before the interviews occurred. The researcher was therefore enabled in asking questions within an appropriate context. For example, by being familiar with an instructor's teaching style the researcher was able to ask specific questions about why the instructor chose to regularly proceed in particular ways. The October time period was also chosen because it gave the instructor opportunity to become comfortable with the researcher attending and observing classes. At the time of the interview the researcher was therefore not a stranger to the instructor.

The interviews were conducted in the instructor's office at a time convenient to the instructor. The instructors were asked all the questions on the Instructor Interview Question sheet, generally in the order they appear on this sheet (see Appendix G). An exception to this order was that the instructor at the university was not asked question 5 under the Teaching Calculus section. Question 5 did not pertain to his situation. The interviews were 60 to 90 minutes in length. They were recorded on audio cassette and transcribed after the interview.

After the interviews were transcribed, transcriptions were given to the corresponding instructor in order that he might clarify or correct anything said he felt was inappropriate or inaccurate. The researcher then used the transcriptions in writing descriptions of each instructional setting. These descriptions can be found in Chapter 4.

**Student Interviews**

Clinical interviews were done with 5 or 6 students in each class. Four complete interview transcripts are in Appendix T as samples of the format of the interviews and the nature of students' responses. The interviews were done to address the first three research areas related to students' *language use, sources of conviction* and manner of construction of conceptualizations. As well, the student interviews aided examination of the fourth research area, the impact of instruction upon student learning. As outlined by Ginsburg (1981), clinical task-based interviews are appropriate for psychological research on mathematical thinking because they allow discovery and identification of cognitive structures and thought patterns, along with evaluation of competence. To accomplish this, Ginsburg outlines the use of open-ended as well as relatively focussed tasks. These tasks include written or verbal mathematical questions to be worked through, and might also include concrete materials to be manipulated. Both types of tasks were included in the problem set for this study. Students were therefore asked to identify, describe, interpret, explain, or apply limit and derivative concepts. The initial written calculus problems (see Appendix H) given to students in the interviews were standardized (i.e. the same for all students), and subsequent written and oral questions were contingent upon previous responses.

The interview interaction was therefore characterized by the researcher using flexibility in the questioning process. This methodology allowed the researcher to probe a student's responses for insights into the student's conceptualizations, *language use* and *sources of conviction*. The probing was done through active listening, encouraging vocalization, asking for clarification, and requesting reflection (Clement & Konold, 1989; also see Appendix I). Silence or repetition of a student's words were also used to probe students' responses. Therefore, a feature of the probing techniques was that they did not harass interviewees, but gave them sufficient time and opportunity to fully answer problems and related questions (Powney & Watts, 1987).

The interviews were primarily focused on students' responses to calculus problems, but also incorporated relevant personal interview questions. The personal information requested included: mathematics background and grades, perceived difficulty of mathematics in high school and in calculus, amount of time spent daily outside class studying calculus, reasons for taking a calculus course, career plans, attitudes towards calculus, perceptions about the ease and difficulty of particular calculus topics, perceptions about the use of language and mathematical notation in learning calculus, ways of determining "correctness", and exposure to calculus in other courses. The set of questions asked of the students can be found in Appendix J. Thus, the interviews provided extensive performance examples and detailed accounts of students working with calculus problems. They formed the data base for analysis of students' *language use* and *sources of conviction.*

The calculus problems of the interviews can be found in Appendix H. This problem set was selected to provide a range of mathematical representations within which students could work, including words, symbols, graphs, and applications. The problem set was also designed to provide opportunities for translation between these various forms of mathematical representation. For example, Problem 12 required students to interpret and translate mathematical language (words and symbols) into a graphical representation.

The interview problems were also selected to include both "skill" and "concept" questions. The primary aim of skill questions (Problems 3a, 7 and 10) was the assessment of a student's basic skills for limit evaluation and differentiation. The remaining problems (concept problems) were aimed at examining students' conceptualizations related to the limit and derivative. These problems required the student to recognize, describe, explain or translate to another representation situations related to limit and derivative concepts. A range of situations within which to examine students' *language use* and *sources of conviction* was therefore provided.

The problem set was ordered so that problems were interspersed with respect to their emphasis on concepts and skills, and emphasis on mathematical representations using words, symbols, graphs or applications. Problems 2, 3a, 3b, and 8 each have two versions because the infinitesimal approach to instruction used at Gamma College uses nonstandard terminology and notation for the limit concept. The second version of these problems was the question given to students at this institution.

Except for Problems 1 and 12, the problem set was selected from a larger set of clinical interview problems that had been field-tested in the pilot study. Problems were eliminated from the larger set for the following reasons: (1) their solutions were too lengthy to incorporate into a one hour interview, (2) they were too difficult for any of the pilot study interview students to begin to answer or to answer completely, (3) their wording or format caused confusion, and (4) they were not found to be useful for probing students' calculus ideas. The rationale for each problem chosen for the final problem set will now be discussed.

Problem 1:

1. A friend of yours who knows nothing about calculus is wondering what it is all about. What would you say to your friend to explain what calculus is all about?

Problem 1 was aimed at examining students' general ideas about what calculus is all about. Their responses to this question were of interest because what students see as the general nature of calculus is likely to reflect their *sources of conviction* related to calculus. For example, a student who sees calculus as a collection of rules and procedures is likely to see rules and authority as a main source of truth. In comparison, a student who grasps underlying principles and purposes is likely to see himself or herself as a source of conviction.

Problem 2:

2. For each of the following sequences of numbers, decide whether the sequence has a limit. If so, what is this number?

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$$

$$3.9, \quad 3.99, \quad 3.999, \quad 3.9999, \quad 3.99999, \quad 3.999999, \ldots$$

(Gamma College)

2. For each of the following sequences of numbers, decide whether the sequence rounds off to a particular number. If so, what is this number?

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$$

$$3.9, \quad 3.99, \quad 3.999, \quad 3.9999, \quad 3.99999, \quad 3.999999, \ldots$$

Problem 2 was designed so that students could work with the limit concept in a situation (sequences) that was familiar to them from pre-calculus (high school) mathematics. The two sequences of this problem were chosen so that their patterns could be easily recognized. The interview was therefore enabled in a focus on the limit concept, rather than determination of a number pattern. Since this latter situation was a difficulty with some of the sequences that had been used in the pilot study, precautions were taken to avoid it in the main study.

One sequence was given in decimal form because unending decimal representations constitute a key component of the infinitesimal approach to instruction used at Gamma College. Students at this institution are introduced to the hyperreal number system, and infinite decimal representations are used as a means of becoming familiar with this system. Both the language and notation used in the hyperreal number system differ from that used in the real number system. For example, the terms "infinitesimal" and "infinite" numbers in the hyperreal number system are used and given representations, respectively, as:

$$i_0 = 0.00\ldots01{,}000\ldots$$

$$I_0 = 1{,}00\ldots0$$

The comma in this notation is used as a marker of $N_0$ decimal places, where $N_0$ is an infinite even integer (the validity of this definition has previously been established at this point in instruction).

The two sequences were also chosen because they differed from each other in that one approached its limit from below, while the other approached its limit from above. It was revealed in the interview responses that a potentially more informative choice in terms of students' conceptualizations of the limit would have been a sequence that actually attained its limit. An example of such a sequence is: 1, 2, 3, 4, 4, 4, 4, ... .

Problem 3:
3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}$$

(b) What does "limit" mean to you?

(Gamma College)
3. (a) Round off the following:

$$\frac{M^4 + 4}{M^3 - M + 5}$$

(b) What does "round off" mean to you?

Problem 3a was designed primarily as a skill question on limit evaluation, a skill emphasized in introductory calculus. The particular limit used was chosen because the value of its limit is not finite. This feature made it different from the previous two limits of Problem 2. It also gave opportunity for the interview to focus on ideas related to infinity and relative size, prevalent notions in calculus.

Problem 3b was aimed at investigation of students' conceptualizations of the limit concept. It was included on the same page as Problem 3a to provide students with a context within which they might choose to answer the question. Problem 2 and a student's written response to it were also made available to the student while he or she responded to Problem 3b. If students did not say anything in relation to limits beyond the context of Problems 2 and 3a, then the researcher probed by asking such questions as: Can you draw a graph or picture of a limit/rounding off situation? Do you have any other examples of limits/rounding off? What is the purpose of taking a limit/rounding off? What else can you say to explain limits/rounding off?

Problem 4:

4. What can you say about the function $y = \dfrac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?

Problem 4 was designed to give students opportunity to apply limits in a situation that did not explicitly ask for limit evaluation. The function of Problem 4 was chosen for its interesting features, including: evaluating its limit as $x$ approaches 2 yields an indeterminate form, its equation might lead one to think the corresponding graph is parabolic or has a vertical asymptote, the limit as $x$ approaches 2 is easily evaluated by factoring the numerator, and the graph is a straight line with a "hole" at $x = 2$.

Problem 5:

5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.

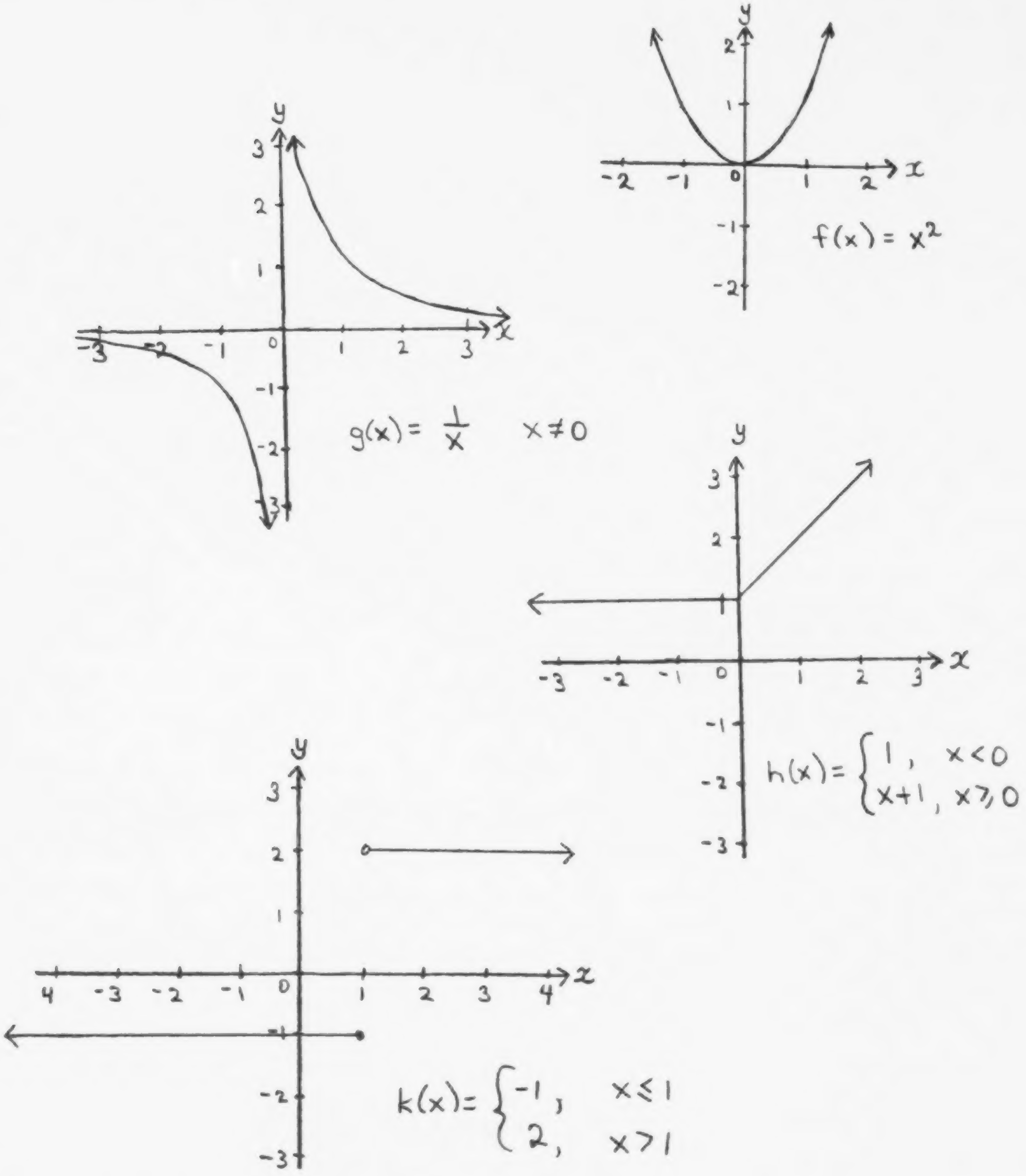


Figure 1. Problem Statement for Clinical Interview Problem 5

Problem 5 was adapted from one of the clinical interview questions used by Orton (1983b). Orton found this question revealed a variety of student ideas related to continuity. The range of responses given by students in the pilot study revealed similar findings. However, the third and fourth functions of the question were altered slightly from the pilot study. The change was made to avoid confusions that arose in distinguishing between the graphs and the axes. Thus, the third and fourth functions were translated away from the origin, and the two segments of the fourth function were translated away from the axes.

Another reason for the choice of Problem 5 was that it provided opportunity for students to work with a limit related concept (continuity) in a visual (graphical) as well as symbolic environment. This also gave the interviewer opportunity to probe aspects of a student's continuity conceptualizations related to physical (visual), verbal, and symbolic representations.

Problem 6:

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?

Problem 6 was similar to Problem 3b in that its purpose was the investigation of students' conceptualizations on a broad level. In this instance the concept was the derivative. The question was intentionally worded to leave it open-ended. Students were therefore able to respond in any context (symbolic, graphical, verbal, or physical). After a student's initial response the researcher asked for examples or explanations of the derivative in contexts the student had not yet given. Thus, each student was specifically asked to explain the derivative verbally, symbolically, graphically and physically (i.e. using a real world application or example).

Problem 7:

7. Find the derivative of each of the following:

$$y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}$$

$$F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7$$

Problem 7 was primarily a skill question and was aimed at determining if a student had mastered the range of derivative rules taught in introductory calculus. The two functions were chosen to require use of all the standard derivative rules (derivative of a constant, derivative of a function multiplied by a constant, power rule, sum rule, product rule, quotient rule and chain rule). The two functions were intentionally technically complex in that taking the derivative of each involved the use of several rules. Students who experienced difficulties handling so many rules simultaneously were given opportunity to work with just a portion of a function. For example, a student was given just the numerator of the first function to differentiate, or just the first parentheses and its exponent of the second function. A student's ideas related to the derivative were not probed in this question, except in relation to the chain rule. Students who spoke of their use of the chain rule were asked to explain why the rule functions as it does.

Problem 8:
8. What interpretations do you have for the expression below?

$$\lim_{h \to 0} = \frac{f(x+h) - f(x)}{h}$$

(Gamma College)
8. What interpretations do you have for the expression below?

$$\frac{dy}{dx} = \frac{F(x+dx) - F(x)}{dx}$$

Problem 8 was aimed at the investigation of students' interpretations of the formal, symbolic definition of the derivative. It was only asked of students whose response to Problem 6 did not include explanation of the derivative in terms of its symbolic definition.

Problem 9:

9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?

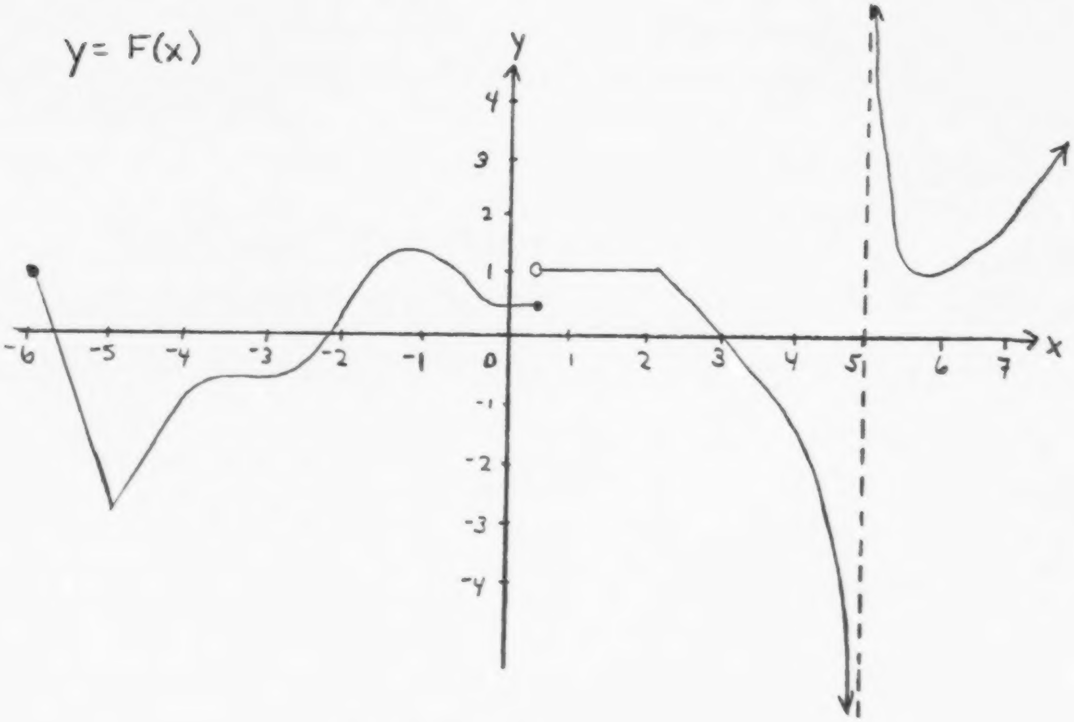


**Figure 2. Problem Statement for Clinical Interview Problem 9**

Problem 9 was designed to explore students' graphical interpretations of the derivative. Of particular interest were their reasons for determination of points where a derivative does not exist. Students who were able to give valid reasons for the derivative not to exist at particular points were then asked to justify their claims algebraically or symbolically.

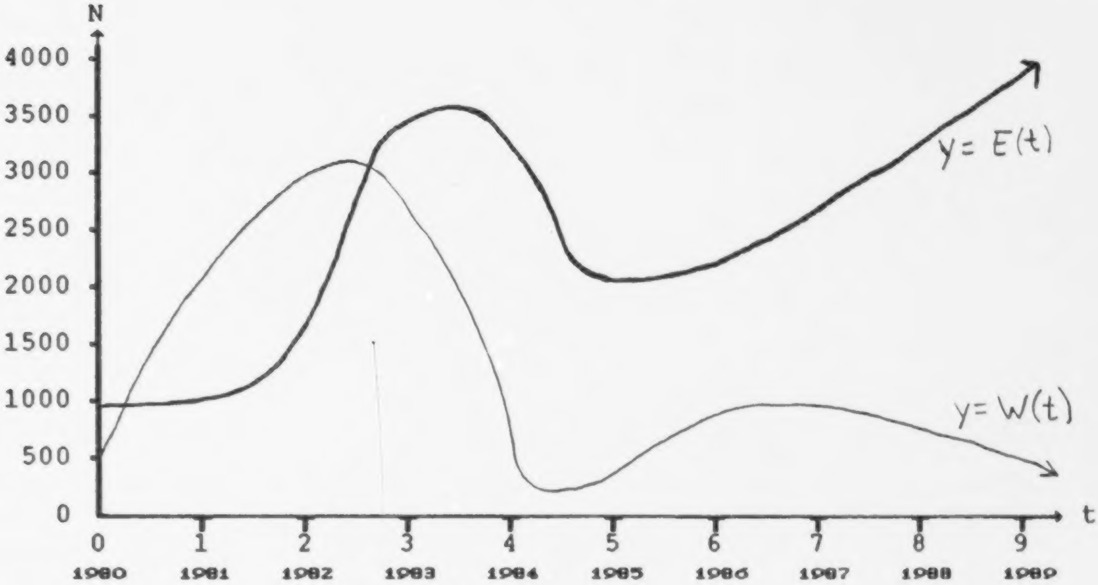
Problem 10:

10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point $(0,-2)$.

Problem 10 was a skill question to assess a student's ability to both recognize a need for and carry out implicit differentiation. A student's conceptualizations of the derivative were not probed in this question, except to determine if he or she connected the notions of slope and derivative.

Problem 11:

11. The number of elk in a national park at the beginning of each year is represented by the function $y = E(t)$ as shown on the graph below. The number of wolves is represented by the function $y = W(t)$, also graphed below.



(a) At what exact point in time was the number of elk increasing most rapidly?

(b) During what time period was the rate of change of the number of elk decreasing?

(c) If you are told that for $0 < t < 4$ (ie. from 1980 to 1984) the equation for $y = g(t)$ is $W(t) = -100t^3 + 1600t + 500$ (t measured in years), how would you determine all critical points of W?

(d) How would you use the critical points found in part (c) to determine the local and global extrema of W?

(e) At what point or points in time is the number of wolves not changing?

**Figure 3. Problem Statement for Clinical Interview Problem 11**

Problem 11 was designed to have students work with the derivative concept in a real world context. Parts (a), (b) and (e) involved concurrent interpretation of words and graphs, while parts (c) and (d) relied on use of mathematical symbols. Parts (c) and (d) were not included as part of the question set after the first three interviews because it was found that the interviews were longer than had been intended and communicated to the student volunteers. Thus, parts (c) and (d) are not included in the results discussed in Chapter 4.

Problem 12:

12. On the axes given below, sketch the graph of a function with the following properties:

    (a) y coordinate of $-3$ when $x = -8$

    (b) derivative of 2 when $x = -5$

    (c) local maximum when $x = -1$

    (d) derivative of 0 when $x = 2$

    (e) slope of 1 when $x = 4$

    (f) when $x = 7$, a point where the function is continuous but not differentiable

    (g) $f'(x) < 0$ and $f''(x) > 0$ when $x > 8$

Problem 12 was designed to investigate students' abilities to interpret and translate into graphical representations mathematical words and symbols related to the derivative. Some students were asked to do part (g) separately on the reverse side of the page without the condition $x > 8$. This alteration was due to insufficient space on the axes provided.

The interviews were done with 5 or 6 students from each class. They were conducted in the last 3 weeks of the school term, after all units on the limit and derivative were completed. The interviewer asked the personal interview questions orally, although students were simultaneously given a typed version of the question set (see Appendix J). The clinical interview problems were given to students typed on separate sheets of paper. The exception to this format was with Problems 3a and 3b. These problems were presented on the same page as explained under the rationale for Problem 3b. Students were asked to respond in writing and orally to the problems. After their initial responses they were asked probing questions (see Appendix I).

Interviews lasted between one and two hours. They were conducted at a time convenient to the student, and were done in a classroom at the student's university or college. They were audio recorded and later transcribed. Students' related written calculations and responses were retained by the researcher as part of the data base. One to two weeks after the initial interview a short follow-up interview was conducted with most of the interview students. Follow-up interviews lasted between 15 and 30 minutes. Their main purpose was to have a student clarify or expand upon responses from the first interview. Frequently during these follow-up interviews the researcher also asked additional clinical or personal interview questions to confirm or deny hypotheses generated from the first interview.

In relation to interviews Powney & Watts (1987) note the following:

> . . . an interview is a contrived social situation with an asymmetrical relationship between the interviewer and interviewee, . . . to some extent all interviews are seen as threatening by those being interviewed (p.44).

It is therefore important that an interviewer establish a good relationship between herself or himself and the interviewee. If the interviewer does not gain the interviewee's confidence, both the validity and reliability of an interviewee's responses will be threatened. Establishment of a good relationship between the researcher and the student interviewees of this study was aided by the fact the students interviewed were all volunteers, and the interviewer was not involved in determination of students' grades. As well, the researcher explained verbally to each class the nature and purpose of the interviews.

Students were told the interviews would involve working through about 12 calculus problems, some similar to exercises they had done in their course, and others aimed at explanation of basic calculus ideas. They were told that the researcher's main interest was to study students learning calculus, and how various factors influence learning. It was emphasized to them that the focus of the interview analysis would not be upon what a student did right or wrong. Rather, the focus would be on what a student did and what his or her reasonings were. Students were reminded that participation in the research would not figure in the determination of grades. They were also told the questions were aimed at gathering information on students' experiences in their calculus courses in terms of their mathematics background, study practices, attitudes towards calculus, and impressions about what either helped or hindered their learning.

It was also explained to students that interviews would be tape recorded and later transcribed, and that their written work would be retained by the researcher. It was emphasized to them that confidentiality would be maintained and their names would be changed in any research reports. The researcher also made clear to students that since she wished to interview a range of students, students achieving at all levels were desired. The researcher repeated her request for volunteers until a sufficient number and range were obtained from each class. However, five of the six interview students at one of the colleges were from a different section of the course than that the researcher observed. This other section was taught by the same instructor, using the same syllabus, textbook and lecture notes.

In order to build rapport with student interviewees before the clinical problem set, the researcher decided to begin the interviews with the personal interview questions. However, this order was reversed after the first interview (with Leanne) because the researcher found it difficult to turn the focus of the interview away from discussion and onto working through problems. The difficulty did not occur once the order was switched. Since the switch encouraged students to already be thinking about calculus when asked the personal interview questions, the change in order gave them opportunity to relate experiences in their course to experiences doing the clinical interview problems.

Consequently, as more interviews were done, the researcher frequently integrated personal interview problems into discussions of the clinical interview exercises. An example of this occurrence is given below. It is taken from the interview with Betty, and integrates the personal interview question related to language into the clinical portion of the interview.

> Betty: Yeah. If it was like ah, some way it would be related to the math. Like they make some words that are totally awkward. . . . It makes it harder. Like this is an endpoint. You understand that. . . . Or this would be like a local maximum, or this one would be a local minimum. Words that make sense.
>
> Interviewer: Well what can you say in general about math, either the terminology or the symbols, or the way things are described? How do you find the language either helps or hinders you? You've just mentioned one thing.
>
> Betty: Another thing is the symbols. For instance, like symbols like this [an integral symbol].
>
> Interviewer: The integral one.
>
> Betty: Yeah. And then symbols like this confuse me [summation notation].
>
> Interviewer: Why do they confuse you?
>
> Betty: Cause like when we have an equation and we have to go like this, like we have to put it in this. I don't understand it.

The increasing integration of personal interview questions and clinical problems does not significantly affect reliability of the interview process because students were asked the same set of questions. Although it allowed the researcher to delve more fully into a student's ideas and experiences, the difference in order was compensated for by the use of follow-up interviews. Components of any interview that had not been pursued adequately by the researcher were given additional time in the follow-up interview.

The only other switch in order from that initially intended was with Problem 1 of the clinical interview problems. The first two students interviewed (Leanne and Sally) found this question difficult to answer. They had very little to say, and this lack of words appeared to make them uneasy. Consequently, to allow students to begin the interviews with what they would perceive as success, Problem 1 was moved to follow Problem 12. When this order was followed for subsequent interviews students appeared to be more comfortable during the initial few minutes of the interview. To allow Leanne and Sally fair opportunity to answer this question they were asked it again in the follow-up interviews.

Before beginning the clinical interview problems the researcher outlined the format that would be followed. Students were asked to respond to the written problems in whatever form they were comfortable with, whether that be writing, talking, writing and talking simultaneously, writing followed by talking or vice versa, or writing and talking interchangeably. They were encouraged to explain what they were doing, and were told the interviewer would interject along the way with questions asking them to clarify or extend what they were doing. The interviewees were also told the interviewer would occasionally give them short problem tasks related to the initial written problem. In addition, it was emphasized to the students that it was both acceptable and worthwhile for them to identify when they were confused by problems, or uncertain as to how to proceed. As well, they were reminded that confidentiality would be maintained.

Analysis of the student interviews was in terms of their *language use* and *sources of conviction*. The researcher initially intended the analysis to proceed according to the dimensions of *language use* and *sources of conviction* outlined below. These dimensions were developed from the pilot study research.

### *Language Use*

1. Context (referring to the circumstances within which language is used)

A. Context Related

Words or symbols used to orient thought or action to features of an environment.

(a) Physical Environment: the referents of the words or symbols are objects or processes of sensory-motor experiences.

(b) Mathematical Environment: the referents of the words or symbols are objects or processes of a system of definitions, axioms, theorems and rules of inference.

B. Context Free

Words or symbols used as rules, independent of features of a physical or mathematical environment.

2. Type (referring to the nature of the language used)

A. *Technical*

Use of "correct" mathematical terminology or symbols in terms of what is accepted by the mathematical community at large.

B. *Everyday*

Use of words or symbols not recognized by the mathematics community for use in unambiguous mathematical discourse.

The following chart outlines possible categories of *language use* according to the above dimensions. Examples of possible *language use* in each category are given after it.

| | | **Technical** | **Everyday** |
|---|---|---|---|
| **Context Free** | | i | ii |
| **Context Related** | Mathematical | iii | iv |
| | Physical | v | vi |

**Figure 4. Language Use Categories**

Category:

(i) A student says a derivative is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ but is unable to relate this to a physical or mathematical context. That is, the student sees this expression as a rule for finding derivatives.

(ii) A student says a derivative is "rate of change", but is unable to explain what a rate of change is.

(iii) A student explains the definition of derivative in terms of measurement of instantaneous rate of change of the values of a function. This is done while using words and symbols such as: $f(x+h)$, average rate of change, or limit as $h$ approaches zero.

(iv) A student explains the definition of the derivative in terms of a rate of change, using words and symbols such as: when you plug in $x$ values right near this value, or how fast the $y$ value changes.

(v) A student draws a graph of a function, marks points, and draws various secant lines to explain how the derivative is the limit of the slopes of a series of secant lines. This is done in relation to words, phrases and symbolic expressions such as: as $x_i$ approaches $x$, point $Q_i$ approaches point $P$, $\frac{f(x_i) - f(x)}{x_i - x}$, or the slope of the secant line will be the ratio of height to base in a right-angled triangle.

(vi) The student draws a graph as in (v), but the words and symbols used are such as: as this point gets closer and closer to here, the secant will move down and its slope will get closer to the slope of the tangent line.

### *Sources of Conviction*

A. External

Textbook, teacher or other individuals, memorized mathematical definitions, rules or algorithms, or no convictions.

B. Internal

Experience of the world, personal knowledge of mathematics, or personal beliefs.

Examples of what a student might say for each of these categories are:

A. External

Textbook: The book did an example almost like this so I followed the format, but I don't know why it works.

Teacher or Other Individual: We learned in the lab to follow this sequence of steps. I'm not sure why, but I know it works.

Definitions, Rules or Algorithms: Because I know that whenever you have a y you have to multiply by dy by dx.

No convictions: I don't know why. I just do it that way.

B. Internal

Experience of the World: As you reach the top of a hill you'll quit climbing up and things will flatten out, so the slope must be zero.

Mathematical Knowledge: If a function is continuous and has a negative value at 2 and a positive value at 3, then it must cross the x-axis between 2 and 3.

Beliefs: I think it has something to do with the way the curve changes the way it bends. I'm not sure of the details, but I know that's the way it goes.

During extensive examination of the interview transcripts of the main study difficulties arose with the reliability of the context category for *language use*, and also with the distinction between external and internal *sources of conviction.* These difficulties are discussed in Chapter 4, along with the analysis scheme that emerged in its place and results of the final analysis.

# 4. RESULTS OF THE STUDY

## A. Introduction

The purpose of this study was to investigate the nature and role of undergraduate calculus students' *language use*, *sources of conviction* and manner of construction of calculus conceptualizations. The investigations were undertaken with students from three different post-secondary institutions. The institutions used three different approaches to calculus instruction: technique-oriented, concepts-first and infinitesimal instruction.

In the first section of this chapter data from the classroom observations, textbooks and exercise assignments, instructor interviews and student questionnaires are used to describe each of the three calculus classes in terms of course formats and content, the nature of the three approaches to instruction, and student backgrounds. The descriptions are aimed at addressing the fourth and fifth areas of inquiry of this study, the nature of instructional events as delivered to students and the potential impact of instruction on student learning. In the next section results from the systemic classroom observation and textbook and exercise assignments analyses are presented. These activities are also aimed at addressing the fourth and fifth areas of inquiry. In the last section findings from the student interviews are presented and discussed. These findings address the first four of the research objectives, the nature and role of students' language use and sources of conviction, students' manner of construction of conceptualizations, and the potential impact of instruction on students' learning. Included in all sections are reports of analysis categories that emerged as data was analyzed.

## B. Research Setting Descriptions

In this section the calculus courses, instructional settings and student backgrounds at the three post-secondary institutions are described. Descriptions are provided for each of the three instructional settings in terms of the content and format of each course, the nature of each instructor's teaching style and practices, and each instructor's impressions about, and perceptions of introductory calculus and introductory calculus students. In the final section, Class Backgrounds, student backgrounds in each class are reported. Students' mathematics and language backgrounds, major subject area, career plans, and attitudes towards mathematics and calculus are reported. This information provides description of the characteristics and similarities of the three calculus classes. The descriptions are based on data collected from observations, curriculum analyses, instructor interviews, and student questionnaires. The data used from the classroom observations are taken from the Other Observations column of the Classroom Observation Sheet (see Appendices C and D), and from specific instructional events recorded during classroom observations. The date of occurrence is indicated whenever specific instructional events are given. Data related to the curriculum are taken from the course outlines, textbook tables of contents, and specific textbook presentations.

The names of the institutions, Alpha University, Beta College and Gamma College, are fictitious. The names of the instructors, Professors Alpha, Beta and Gamma, are also pseudonyms. However, the people, places and situations are real, and care was taken to describe them accurately. For reasons of confidentiality, the textbooks at the colleges and the lab manual at the university are not specifically named.

### Introductory Calculus Instruction at Alpha University

Alpha University is a large urban university enrolling approximately 25,000 full-time students. The introductory calculus course at this university is representative of such courses across North America. The instructor of the research class studied at the university, Professor Alpha, said this about his course:

> The sorts of things that are taught and the order which they are taught is pretty well set and has been for years. And that's not just here. That's essentially all across North America. That this course is pretty much identical to calculus courses offered to the general audience, science students say, in universities all across Canada.

A standard textbook is used for this course, Single Variable Calculus (J. Stewart, Brooks/Cole Publishing Company, 1987). The content in this text covered in the course includes: limits, introduction to the derivative, derivative rules, applications of the derivative, the definite integral, techniques of integration, and applications of the integral. The exact sections of the textbook that are covered in the course are in Appendix K. The approach used at the university for introductory calculus instruction is "traditional" in that emphasis is put on learning techniques for differentiation, integration, graphing, and the solution of problems. Much class time and textbook space is devoted to methodically providing examples of such techniques. Concepts in the textbook are generally briefly introduced intuitively or informally, then followed by precise definitions, related theorems and proofs, and numerous example problem solutions. More details on the textbook can be found in a later section of this chapter. The approach to instruction used at Alpha University will be referred to as "technique-oriented" instruction.

The calculus course at Alpha University is structured so that students meet with their instructor for three 50 minute lecture hours per week. Professor Alpha's class was in the mid-afternoon, and had an enrollment at the beginning of the school term of about 100 students. Students also have a fourth 50 minute lab session conducted by a graduate student in mathematics. Students are enrolled in one of several lab sections, each lab section containing students from more than one lecture section. Enrollment in lab sections is restricted to 25 to allow opportunity for students to ask questions of the lab instructor.

Labs were observed by the researcher on five occasions. Each lab observed was conducted by a different individual, and for each of these labs the native language of the lab instructor was not English. The lab instructors' verbal proficiency in English varied, ranging from fluent and clearly spoken English, to communication that was hesitant and spoken with an accent the researcher found difficult to understand. Labs were conducted in regular classrooms, while lectures were taught in a large, tiered lecture theater with a seating capacity of about 200.

The format of each lab observed was identical. During the first 20 to 30 minutes the lab instructor worked through several lab manual exercises on the board. These were usually questions for which students requested solutions, but also included questions chosen by the lab instructor. Following the sample exercise solutions, the last 20 to 30 minutes was spent completing a weekly lab quiz that tested competence with calculus skills. The quizzes were created by the individual lab instructors. Students also completed a one hour lab exam in the last lab period of the term.

On a weekly basis students in the course were also given assignments to be completed. These were graded by a marker (a graduate student in this case) and returned to the students. Students completed a one hour midterm exam in late October, and a two hour final exam in December. Final grades were determined as follows:

| | |
|---|---|
| Final Exam | 50% |
| Midterm Exam | 25% |
| Exercise Assignments | 10% |
| Lab | 15% |

The goals of introductory calculus at Alpha University, as interpreted by Professor Alpha, focus on students' development of problem solving capacities. Professor Alpha says the course aims at students learning to "use the central ideas of calculus to solve problems." He said the following with respect to the course:

> Problem solving is the most important thing. That they can take information presented in the form of a few essential ideas and apply it to a problem that is presented to them that they have not seen before. Using perhaps techniques that are similar to ones they have seen before. Ah, but if they can solve new problems using the ideas, then I would be ecstatic.

Professor Alpha also noted that an aim of introductory calculus is that students learn some of the essential ideas of calculus. In the following extract from the interview with him, Professor Alpha outlines what he considers are these essential ideas:

> I would hope that they would remember the definition of the derivative, what goes into it in terms of graphing, perhaps using the derivative and tangent line. I hope they'd remember something like the Fundamental Theorem of Calculus. And its central role in connecting integration and differentiation. And the way it is used to solve integration problems.

In relation to the introductory calculus course, Professor Alpha also said he has always been "bothered" by the high failure rate. He sees "basic techniques in algebra and trigonometry, and to some extent geometry", as essential to mastering calculus, yet finds many students entering university are weak in these areas. He also sees calculus as a "challenging course", and "quite different from . . . anything encountered in previous experience in the educational setting." However, Professor Alpha is also of the viewpoint that students can succeed in calculus if they recognize they face a challenge that requires them to "buckle down" and "avail themselves of the various resources that are offered". He stated this to the class at the start of the term, and also told them he believes "most deeply" that "anybody who is talented enough to get into university is talented enough to pass that course [calculus]." Thus, the fact that so many people find calculus so difficult is a concern to Professor Alpha.

Professor Alpha has had over 20 years teaching experience at the post-secondary level. During that time he has taught a variety of undergraduate calculus and algebra courses, as well as a variety of graduate mathematics courses. Professor Alpha says he very much enjoys teaching and completed a doctoral degree in mathematics because he wanted to teach at a university. Although he doesn't always enjoy the "inevitable" "administrative paper shuffling" required in teaching, he enjoys interactions with young people. In relation to this interaction he commented: "Yeah, it's fun. It's fun." Professor Alpha also commented that teaching is rewarding. He finds "delight" in experiences such as the one he described as follows:

> I just enjoy finding out every once in awhile that somebody that you're teaching likes what he sees and is excited by the ideas he's confronted. Is actually learning something new and relevant to his or her life. And is excited by it. Those are very special times and you don't run across them every time a student walks into your office.

Professor Alpha's lectures were very organized, with pre-determined content and ordering. His chalkboard notes were neat, written in a logical order, included titles for topics, definitions, examples, theorems and proofs, and included underlining of key terms in definitions. Professor Alpha's presentations were mathematically "elegant" in that they presented ideas concisely, using correct mathematical terminology and notation. He generally presented ideas in a general form and followed this with specific examples. For example, the concepts of maxima and minima were introduced with a statement of a definition and followed by specific examples, as seen in the following researcher observation notes:

> (October 22)
>
> Maxima and Minima
>
> Defn. We say the function f takes an absolute maximum at $c$ if $f(c) > f(x)$ for all $x$ in the domain of f.
> Likewise absolute minima.
>
> Examples
> (a) $f(x) = x^2$ has an absolute minimum at 0.
>
> *It takes its smallest value at zero.*
>
> (b) $f(x) = \sin x$ has an absolute maximum at $\frac{\pi}{2}$
>
> *Note the point at which it takes its absolute maximum does not have to be unique. . . . lots of points . . . just has to take its greatest value.*
>
> Defn. We say f has a local maximum at $c$ if there is some open interval I about $c$ such that $f(c) > f(x)$ for all $x$ in I.
>
> *In some interval about c, f at c is largest.*
> *What we do for maximums we can do for minimums.*
>
> Likewise local minimum.
>
> *Let me draw a function.*

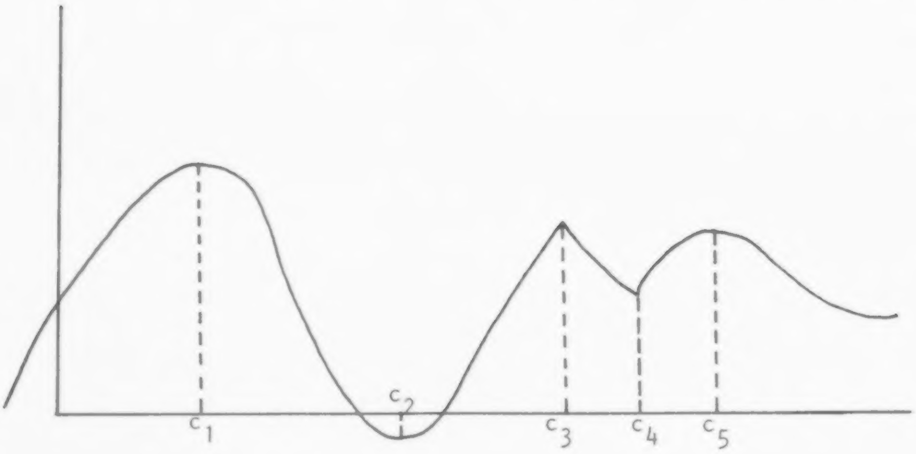


**Figure 5. Graph 1 Drawn by Professor Alpha on October 22**

> *Various points on the graph are of interest in this discussion. Here's one. Here's another. Where is the absolute maximum for this function?*

Professor Alpha regularly proved theorems and often did so by proceeding logically and algebraically to verify the conditions and conclusions of a theorem. For example, proofs of derivative rules were done this way. At other times Professor Alpha demonstrated graphically or algebraically the plausibility of a theorem. An example of this is seen in the following excerpt from researcher observation notes:

> (October 22)
> Theorem If f takes a local maximum at c and f'(c) exists, then
>
> *Can you tell me anything?*
>
> *- It will equal its value at y. At that point it's a line right across.*
>
> *You mean horizontal.*
>
> *- If it's concave up or down at the place where it changes that could be a max or min.*
>
> *If the derivative is positive then the tangent looks like . . .*

Figure 6. Graph 2 Drawn by Professor Alpha on October 22

*. . . then the graph has positive slope.*
*. . . then there are going to be points to the right that have larger values.*
*If the tangent has negative slope then . . .*

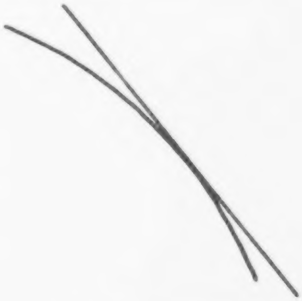


Figure 7. Graph 3 Drawn by Professor Alpha on October 22

*Our intuitive idea that the tangent line has to be horizontal is correct.*
then $f'(c) = 0$.

Professor Alpha frequently did not speak while writing on the chalkboard, writing definitions or statements of theorems, or working through the calculations of an example. In addition, although he generally used precise terminology for both his written and oral presentations, he also sometimes gave alternative verbal descriptions. This can be seen in the above excerpt from observation notes, as well as in the following excerpt:

(October 29)

Rolle's Theorem
Suppose
(a) $f$ is continuous on $[a,b]$.
(b) $f$ is differentiable in $(a,b)$
(c) $f(a) = f(b)$

*Let's see if we can arrive at the result intuitively. . . . without lifting the chalk. . . .*
*tangent line everywhere.*
*The graph is smooth. No sharp corners or cusps.*

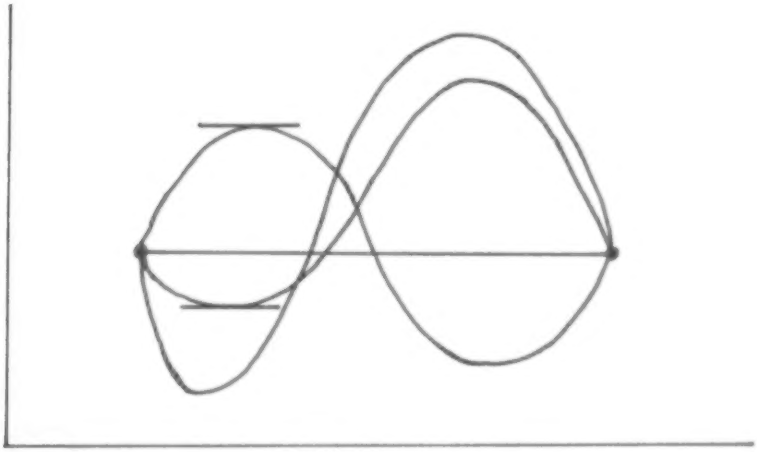


**Figure 8. Graph Drawn by Professor Alpha on September 29**

*Can you see any common features?*

*- There's a value in between.*

*I start out by going up. Can't do that indefinitely. There has to be a turnaround point. What is true there?*

*- Tangent line is horizontal.*

*Or to say it another way, the derivative is zero. Let's check. It's flat here.*

Then for some c on (a,b), $f'(c) = 0$.

In addition to demonstrating how Professor Alpha did not exclusively use precise mathematical terminology, this excerpt also demonstrates how at some point in his lectures Professor Alpha generally incorporated student responses. Other examples of the occurrence of students' responses arose from the following questions being asked of students during lectures. Each example is taken from a different lecture, with the date given in brackets:

(September 12)
*Any thoughts, say geometrically, on what sort of condition we can impose?*

(September 17)
*The -4 is a shifting operation and the -2 is a shifting operation. So what's left?*

(October 2)
*We know we have to manipulate or in some way work on the situation to make it simpler, or to see it in another way. Any suggestions?*

(November 2)
*What do we know about $f$?*

When working through several examples consecutively Professor Alpha generally began with a simple example, then proceeded with examples requiring more algebraic manipulation, use of more than one concept or rule, or use of a different type of function. For example, the sequence of examples used in demonstrating the chain rule was the following:

(October 10)

(a) $y = (x + 1)^3$

(b) $y = [f(x)]^n$

(c) $y = \sqrt{x^3 + 3x^2}$

(d) $y = [x^2 + (x^3 + 1)^{1/4}]^{1/3}$

(e) $y = \sin^4 x$

(f) $y = \sin (x^4)$

(g) $y = \sin [f(x)]$

The interview with Professor Alpha revealed the motivations behind many of the instructional approaches the researcher had observed in his lectures. To begin with, Professor Alpha explained his strategies for teaching large classes, and his impressions of this teaching. He said he does not object to teaching large classes, but says one must recognize a distinct difference from teaching smaller classes of less than 40. Major differences he noted related to the extent of interaction between the instructor and students, and how this constrains the form of the lecture. In large classes "it's sort of like theater". "You react to and respond to the audience as a whole, rather than to particular individuals.

And there's less class discussion, and less give and take." In comparison, Professor Alpha saw the situation in smaller classes to be the following:

> Well there's no question about the fact that the smaller class is the more ideal vehicle for student learning because it's part and parcel of what you're trying to teach. And you're teaching to get the student involved with the material and to get him to confront the material. And in a large class you can't do that, or you can't be sure you're doing that in class. You can sort of present the material and hope that you are good in the sort of technical aspects of presenting the material. You write clearly, speak clearly, and you write the points as clearly and concisely as you can.

Thus, in smaller classes "you have more opportunity to push them [students] into confronting the material right there in class." To compensate for limitations inherent in teaching a large class Professor Alpha said he tries to incorporate a bit of "give and take" into each lecture. He said: "I throw something open. Like what do we do next, or how do we handle the situation?"

In discussing his approach to teaching calculus, Professor Alpha mentioned several aspects of his teaching that have already been outlined from the researcher observation notes. He said he tries to be as precise as he can with his language use during lectures. However, he said he feels students do not need to be that precise. Rather, they have to be "good users of the English language, to write mathematics effectively and communicate what they're doing." He also said he feels it is not necessary for general calculus students to be proficient with mathematical proofs. Professor Alpha noted that mathematical proofs were not emphasized on his examinations for general calculus students. Rather, he said he emphasizes that students "have a feeling for the connection between different ideas." He was of the viewpoint that, for students for whom it would be important, there would be plenty of opportunities in later mathematics courses to develop skills in precise, logical mathematical writing and proof construction. The amount of material covered in an introductory calculus course does not make it possible to emphasize these skills. As a consequence, Professor Alpha believes the primary emphasis of introductory calculus "has to be upon getting them [students] to confront the basic problem solving techniques."

Professor Alpha finds that although he has regular office hours, most students do not come to his office. He finds, as had been observed by the researcher, a few students "will stay after class and ask some questions." Professor Alpha stated he deals in the following way with problems students bring to him:

> And my usual reaction is to do a part of it, or to partly lead them part way into a solution. And then tell them to go see if they can do it that way. See if they can work it out for themselves.

Professor Alpha believes students in his calculus course are primarily motivated by "the grade they get at the end of the course." However, he said he "can't argue with that" because to a certain extent the grade one receives is important. He finds that a "typical sort of mindset" students have is "they want you to tell them how to do it so they can do the exercises and get a good grade for the course." "They don't want to talk about the great ideas in the history of civilization [calculus]." Professor Alpha noted a possible reason for this:

> That's maybe partly because there isn't time to do that and get them to confront all the actual problem solving things that they have to master.

Professor Alpha described learning as a process of "confronting problems". His related notions are clarified in the following:

> [Students] are learning to solve problems logically with the aid of what they know, by confronting problems. And I don't think there's any way of learning anything except by doing. I think that the primary learning that happens in this course happens when they sit down to do an exercise set, or when they start preparing themselves for an exam. And then they sit down and actually take time to confront a situation which they have not confronted before.

In relation to these notions, Professor Alpha said he perceives introductory calculus students as self-reliant in their learning because large classes require that they learn to be so. Students "learn to be self-reliant in the process of getting through the course." "They learn to do things for themselves with the help of whatever they can pick up in the lectures, the book, the lab." In this way Professor Alpha sees learning as a process that arises when students "take what they have seen . . . and try to apply it to a situation that isn't laid out." He said they then learn the following:

> . . . to assign priorities to the information they are given. They learn to organize their thinking so that they can start with the first things and arrive at conclusions. So I think that they learn, hopefully, to think more efficiently.

In summary, introductory calculus at Alpha University is representative of the technique-oriented format of instruction of many introductory calculus courses across North America. Students attend 3 lecture periods per week and 1 laboratory period. They are required to complete weekly exercise assignments and lab quizzes, and are also graded on a midterm and a final exam. Professor Alpha's lectures were organized, and clearly and logically presented. Definitions, concepts, examples, theorems and proofs were clearly identified and presented in a mathematically elegant and logical format. Professor Alpha feels students learn calculus "by doing" and by "confronting problems" and situations "they have not confronted before." He therefore sees problem solving skills as a key goal of introductory calculus.

### Introductory Calculus Instruction at Beta College

Beta College is a small college enrolling about 800 students. It is located in an agricultural region in a town of approximately 13,000 people. Beta College uses what will be called a "concepts-first" approach to introductory calculus instruction. This is instruction in which concepts are explored intuitively before introduction of their formal definitions and proofs, and before skill development is emphasized. This intuitive exploration uses the following approach: (1) experimentation with subcases of a concept, (2) examination of numerical, geometrical, or graphical representations of simple examples of a concept, and (3) formation of analogies to concepts students have already been taught. Rules for specific skills are then presented, including demonstration of their plausibility. Lastly, concepts are revisited and developed in their precise, logically derived forms. For example, the derivative concept is developed as follows: (1) secant and tangent lines on simple functions are examined numerically and graphically, (2) analogies are made between slopes of lines and functions and rates of change, (3) derivative rules are introduced, justified by demonstration of the differentiation process with a specific function, and practiced with simple examples, and (4) the precise definition of the derivative is given and related theorems proved rigorously.

The concepts-first approach to instruction used at Beta College was designed by instructors at Beta College as a response to a high withdrawal and failure rate in standard approaches to teaching introductory calculus. They believed, as noted by Professor Beta, "there might be students who could handle the course and do well if they got the right start." Thus, it was decided to "revamp" the curriculum "to start out with dealing with the topics informally." According to Professor Beta, the belief was that dealing with calculus notions in an informal way first would give students an intuitive feel for concepts before going onto "the precise approach where you use definitions and proofs, theorems and all the rest." Thus, Professor Beta perceives a strength in a concepts-first approach to instruction is that it provides students with a "basis for giving meaning to concepts." That is, he perceives that beginning calculus with an informal, non-rigorous approach allows students to develop the "conceptual basis" for understanding more rigorous approaches.

The document used as a textbook at Beta College is unpublished. It was conceived of and written by the calculus instructors at Beta College so that it would match the concepts-first approach to instruction they wished to use. Thus, it differs from the texts at both Alpha University and Gamma College in the ordering of units. A spiral approach is used, with topics returned to more than once to be developed in further or different ways. The topics covered and the order in which they are covered can be found in the extracts from the text's table of contents (see Appendix K). However, the topics covered are the same as those in standard calculus courses in that they focus on limits, derivatives and integrals, and related skills and applications. At the time this research study was conducted Beta College was beginning a second school year of use of the new textbook.

The calculus course at Beta College is structured so that students meet with the instructor for four 50 minute lecture hours per week. Professor Beta's class had about 40 students at the start of the school term, and was held in the early afternoon in a regular classroom. A fifth optional tutorial hour was also held each week. On a weekly basis students are given exercise assignments additional to exercises available to them in the textbook. Assignment solutions are posted and then assignments are handed in to be checked as to whether or not they have been completed. Professor Beta said that although students are encouraged not to just copy assignment solutions, the instructors realize there will always be students who do just that. However, since assignments form only 10% of a student's final grade, the situation is not deemed by Professor Beta to be unfair to students who do put the effort in for themselves. That is because students who make such efforts will benefit in learning concepts and skills important for success on exams. Professor Beta stated the following with respect to this point: "But they won't get anything out of it if they don't try and do the problems, and expend a lot of effort thinking about them. They won't get much out of them."

In addition to weekly assignments, a student's final grade was formed from four term exams and a two hour final exam. Final grades were determined as follows:

| | |
|---|---|
| Assignments | 10% |
| Term Exams (four in total) | 48% |
| Final Exam | 42% |

Professor Beta perceives the format of several term exams rather than one midterm exam to be a strength of the course. This gives students early feedback as to how well they are handling calculus, and thereby gives them a chance to adjust psychologically and make the "transition from high school" to college. In addition, Professor Beta believes the format of several tests helps students "learn how to handle something like calculus." "The rigor and all the work" expected in introductory calculus is something students must learn to manage in order to be successful in the course.

Professor Beta views the workload in introductory calculus, and in particular in a concepts-first approach to instruction, as a weakness in the course. It demands much of both students and instructors. Since "it's hard to get through all the course material", an instructor can become frustrated. It becomes difficult to teach everything effectively. According to Professor Beta:

> But there's a frustration because it seems like it's hard to teach it all and get them to be able to handle all that within a three and a half month period. . . . But it is a frustration trying to get it all in and get them to learn all of the concepts.

As a consequence of the time constraints, Professor Beta believes it is important for calculus students to learn to take responsibility for their own learning. He finds many students "would like to have the teacher telling them how to do things", but feels that "students who cannot take some responsibility for their own learning just do not make it. They end up dropping out or failing." He has had many calculus students who have not "been able to develop that self-reliance, so they haven't been successful in the course." However, he has also had many students whose self-reliance has built and "become stronger and stronger" as the term goes on. In relation to this point he said:

> That's a rewarding thing to see too because I see that as accomplishing one of our major general objectives which is beyond mathematics, and is to develop self-reliance in pursuing something.

Within the mathematical realm, the concepts Professor Beta wants students to learn from introductory calculus are "the kinds of things that are usually expected in an introductory calculus course." According to Professor Beta, these things include: a basic understanding of the derivative, antiderivative, indefinite and definite integral; an ability to use differentiation on various types of functions; an ability to determine areas through the use of antiderivatives; and an ability to do a variety of related rates, maximum and minimum, and other applied problems.

Professor Beta has had over 25 years teaching experience, including experience teaching mathematics at both the junior and senior high school levels, and at the college level. He said that although teaching involves "a fair bit of keeping of records and things" that has to be done, he enjoys teaching. He expanded upon his enjoyment of teaching in the following excerpts from the interview with him:

> Well, I've been here for 25 years. That says something about what I think of teaching here I guess. I wouldn't stay if I didn't enjoy it. . . . I enjoy teaching here because of the relationship with the students I guess. . . . And I enjoy that because I do get to know all my students, and I enjoy doing that.

> I enjoy the subject matter, mathematics. And mathematics problems. . . . But there's also a challenge in seeing how students think about these problems and whether they can um, and then there is maybe a shift. It shifts from being interested in mathematical problems in themselves to being interested in how students understand mathematical problems. And that's intriguing even when you know all about the problem yourself. So I encounter lots of new problems. Then I also enjoy seeing how students think about these problems.

Professor Beta's lectures followed a pre-determined plan as to content and sequence. His chalkboard notes were organized by titles for topics, definitions, examples, theorems and proofs. Throughout the term his lectures varied in terms of the format followed and the level of rigor used to present ideas. In some instances ideas were presented informally, through use of graphical interpretations or specific symbolic examples. At other times theorems or general forms for rules were given, verified by logical rigorous proofs, and exemplified through specific algebraic or graphical examples. The level of rigor increased throughout the term, as the nature of the corresponding chapters in the textbook also changed. As already outlined, the textbook, and hence instruction, followed a spiral approach. Topics were presented more than once, with later presentations displaying more formal logic and mathematical rigor.

One aspect of the concepts-first approach to instruction that struck the researcher was the number of calculus concepts students were exposed to within the first month of the term. Within the first week students were "doing" calculus. By the end of September, they had worked with the definition of the derivative, all the derivative rules, maximum and minimum problems, and related rates problems. The other two research classes did not begin units related to the derivative until October. The course at Alpha University (technique-oriented instruction) began with review of some high school algebra and trigonometry, and then proceeded to a unit on limits. At Gamma College (infinitesimal instruction) the course began with review and then included a unit to develop and apply the hyperreal number system.

As Professor Beta introduced formal proofs into lectures he continued to incorporate informal interpretations of concepts. This was often through graphical interpretations of ideas. The following three excerpts from researcher observation notes display the increasing level of formality used in lectures, as well as the continued use of informal interpretations (in the third excerpt). All three extracts deal with the concept of continuity.

(September 11)

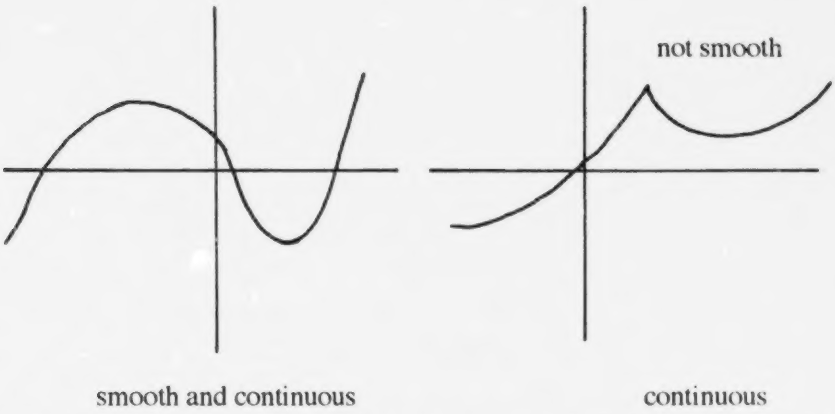


**Figure 9. Graphs Drawn by Professor Beta on September 11**

*No gaps. You could draw it without lifting your pencil off the paper.*
*There's a jump. It's got what we call a discontinuity.*
*Basically it's got a discontinuity if you have to lift your pencil off the page.*

(October 23)

Theorem: If f(x) and g(x) are continuous at $x = a$, then so are the following:

(a) $f(x) \pm g(x)$

*To translate that into English, . . .*

(b) $cf(x)$ where c is a constant

(c) $f(x)g(x)$

*That is, the product of two continuous functions is continuous.*

(d) $f(x)/g(x)$ if $g(a) \neq 0$

*. . . have to add something there.*
*. . . going to prove the last one only.*

(October 30)

From the left

$$\lim_{x \to a^-} f(x) = f(a)$$

*Couple of things implied . . . f defined . . . left-hand limit exists . . . and are the same thing.*

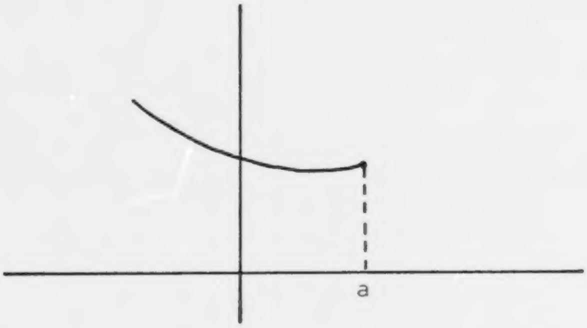


**Figure 10. Graph 1 Drawn by Professor Beta on October 30**

*Doesn't matter what happens to the right of a here.*
*. . . as approaches from the left . . .*

From the right

$$\lim_{x \to a^+} f(x) = f(a)$$

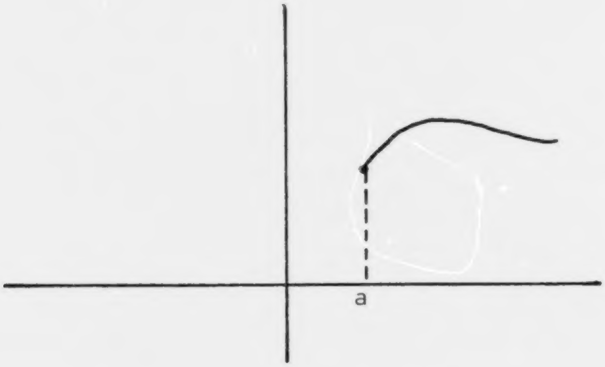


**Figure 11. Graph 2 Drawn by Professor Beta on October 30**

*Has to be defined right at a . . . has to have a value as you come at it from the right*
*. . . has to reach that height.*

All three of the above excerpts demonstrate a common feature of spoken aspects of Professor Beta's lectures. Professor Beta spoke as he proceeded through presentations of concepts, definitions, examples, theorems or proofs. He continually explained orally what he was writing or drawing and why he was proceeding in certain ways. In this way, Professor Beta's lectures explicitly made connections amongst symbolic, verbal and graphical mathematical representations. This feature is further demonstrated in the following excerpt from a lecture on horizontal asymptotes. In this excerpt Professor Beta combines graphical, symbolic and verbal interpretations of horizontal asymptotes.

(October 9)

Horizontal Asymptotes (H-Asymptote)

*. . . should be familiar from high school and hyperbolas . . .*
*lines that a graph approaches as x gets large or x approaches some number.*

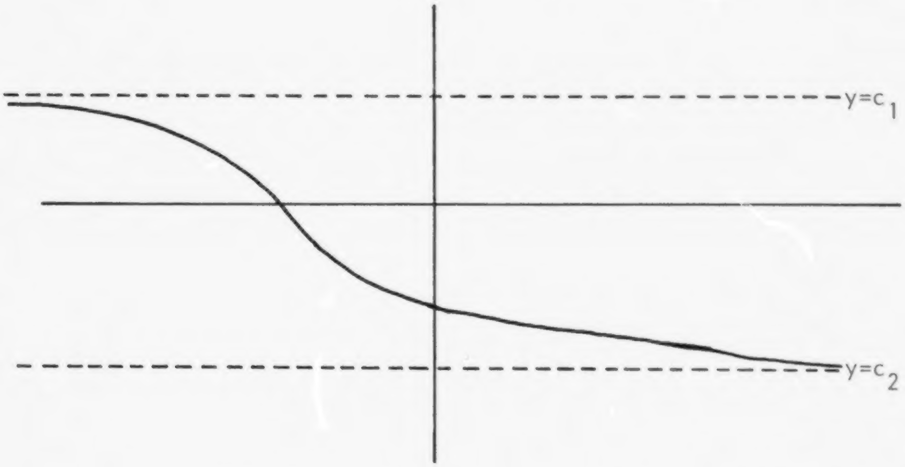


**Figure 12. Graph Drawn by Professor Beta on October 9**

*Concerned about behaviour at extremes way out to the left and way out to the right.*
*. . . as x approaches negative infinity, y approaches . . .*
*. . . as x approaches positive infinity y approaches . . .*

y = f(x) has a H-asymptote $y = c_1$ on the left

*Now you could tell by looking at the graph.*
*Sometimes it will be the other way round . . . find the horizontal asymptote.*
*Way we would find that out is . . .*

$$\lim_{x \to -\infty} f(x) = c_1$$

*as x approaches a large negative number . . .*

y = f (x) has a H-asymptote $y = c_2$ on the right

$$\lim_{x \to +\infty} f(x) = c_2$$

*Can see by looking at the graph.*
*Again, you need to find them to visualize what g will look like.*
(instructor drew a graph at this point, but observer did not have time to record it)
*Does it have an H-asymptote on the left or on the right?*
*What are they? Does it come from below or above? Like that or like that?*
*Tell by analyzing the limit.*

Continuing on in the same lesson, Professor Beta worked through the following examples to discuss and determine horizontal asymptotes:

(October 9)

$$\lim_{x \to \infty} g(x) = 4$$

$$\lim_{x \to -\infty} k(x) = \infty$$

$$f(x) = \frac{1}{x} + 3$$

$$g(x) = 2x^2 - 8x + 11$$

$$f(x) = \frac{4\sqrt[3]{x} - 1000}{500 + \sqrt[3]{x}}$$

$$k(x) = \frac{2x^2 + 3x - 1}{x^2 - 5x + 2}$$

These examples demonstrate how Professor Beta generally sequenced examples related to a particular concept so as to increase the level of algebraic manipulation required to reach a solution.

The extracts from Professor Beta's lectures display a mixture of technical and everyday language. When asked of his impressions of the role of language in learning calculus Professor Beta said he sees language as an important influence on learning. He said that he teaches by trying to put things into language students can understand. He explained his reasons for this approach as follows:

> And then of course math is like other subjects. You have your own terms and definitions, and that's essential. But one has to start by using the kind of language students can understand. And the same with notation I guess. Notation's . . . very powerful, but one needs to try to introduce it in a way that they will understand. . . . Yeah, language is very important.

Professor Beta also said he finds many students do not use language well, and therefore do not communicate clearly. He sees this situation as unfortunate in that he suspects an inability to communicate well might be a reason for experiencing difficulty in learning mathematics, or vice versa. Professor Beta also wondered if cultural entities such as television might influence the way an individual approaches his or her learning. Specifically, he said:

> I wonder if they're getting used to watching television where the thinking is done for you. You can't do mathematics that way. You have to get at it and do the stuff yourself. And be prepared to do some hard thinking.

In addition to learning by "doing" Professor Beta believes students learn in a variety of ways, and that it's important to keep this in mind when teaching. When asked "How do you think students learn?" his response included:

> How do they learn? That's a big question. How do they learn? I guess I've probably had some assumptions as a teacher about how they learn. I've always thought in the past that if you present things in a very logical manner and show all the steps in the logic then most students can understand. And it's from those gaps in the logic that they don't understand. I realize that, from studying how students learn mathematics, that there are a variety of ways that they learn new concepts. . . . So I think there are lots of ways that they think about problems. And I guess I'm now more sensitive than I was a few years ago in trying to stop and understand how students are thinking about a problem. Instead of just providing the method that I like best. Some students think spatially. They like to have pictures and diagrams, and things like that. Other students don't need that. They think in other ways. But I do still think that in mathematics, basic to all that is it is logical and you need to somehow present the logical stepping stones in the argument for them to understand it. Along with other kinds of things. Diagrams and so on.

In relation to these views Professor Beta views proofs as an important aspect of mathematics learning. He believes it is important that calculus students are exposed to both informal and formal means of justifying mathematical statements. He said: "If students are really going to appreciate what mathematics is all about then they have to have exposure to proof." The reason for this need is the following:

> And I think sometimes the reason of a proof escapes students. . . . Then they wonder why they're doing this proof when it looks like the thing is true without a proof. And so I guess we need to do some work in showing students why proof is necessary, by showing them that our intuition can lead us astray in some cases. And I try to do that on occasion. Just to show them that what seems obvious may not be true. To show them you have to be careful about proving things carefully and logically.

Professor Beta believes calculus students are motivated in a variety of ways. Some of them have "extrinsic motivation" in that they "need it [calculus] for a particular program they're going into, . . . so they're desperately trying to get whatever mark they need." Others are more "intrinsically interested in the subject matter" and are prepared to "struggle with the concepts and work at developing their problem solving skills." This notion of struggling was seen by Professor Beta to relate to learning calculus in the following way:

> So if a student is not prepared to work in the sense of actually thinking. If they're just thinking that things can be, they can be spoon-fed, the ideas can be given, presented to them in a way that they don't have to struggle with, they're never going to manage. . . . Cause that's the nature of problem solving. Struggle.

Professor Beta believes students begin calculus with a good "mathematical repertoire" from high school, but they haven't previously learned how to solve problems. In addition, he said that although they've been given a good background in algebra, trigonometry and geometry, they tend to either forget many things or do not have any conceptual learning behind the skills they have learned. However, he noted that calculus requires students to learn skills such as algebraic manipulation, even if they have not adequately learned them before.

In summary, introductory calculus at Beta College uses a concepts-first approach to instruction in which concepts are examined intuitively before formal definitions and proofs are introduced, and before skill development is emphasized. The course uses a textbook that was written by the instructors at this institution to support a concepts-first instructional approach. That is, the textbook first presents ideas informally, following later with sections emphasizing skill development and formal, precise definitions and proofs. Students in introductory calculus at Beta College attend 4 lecture periods per week. They complete weekly exercise assignments for which solutions have previously been posted, and are also graded on 4 term tests and a final exam.

Professor Beta's lectures varied in format and the level of rigor incorporated, with an increased presence of formal and rigorous presentations as the term progressed. Throughout the term he incorporated informal and graphical interpretations of concepts. Further, Professor Beta explicitly made connections between various aspects of his presentation, orally explaining what he was writing on the board and why he proceeded in certain ways. Professor Beta sees language as important to mathematics learning because mathematics learning requires an ability to think and communicate. He therefore sees "hard thinking", "struggle" and doing the work for "yourself" as essential aspects of calculus learning.

**Introductory Calculus Instruction at Gamma College**

Gamma College is a small urban college enrolling about 2000 students. The introductory calculus course at this college uses what will be called an "infinitesimal" approach to calculus instruction. This approach develops concepts intuitively while using methods related to nonstandard analysis as analytic and computational tools. Infinitesimal methods are the tools by which Newton and Leibniz first developed calculus in the late 1600's. Newton and Leibniz did not precisely define infinitesimal numbers, demonstrate their algebraic properties, nor logically validate computations made with them. As a result, many mathematicians saw infinitesimal methods as a source of unsoundness in the foundations of calculus. This was resolved in the late 1800's when Weierstrass rigorously re-developed calculus using real number concepts. Since then, real analysis methods have dominated the teaching of calculus.

In the 1960's Abraham Robinson used a logically rigorous approach and re-developed calculus in terms of infinitesimal number concepts (Robinson, 1966). Based upon mathematical logic, Robinson's treatise (called nonstandard analysis) is beyond the capabilities of most undergraduate students. However, his discoveries can be translated to a level suitable for introductory calculus instruction. This is done by introducing students to infinitesimals intuitively, then using these numbers to develop calculus concepts in both intuitive and formal ways.

The best way to demonstrate how this approach differs from the use of methods in real analysis, and in particular, from technique-oriented and concepts-first approaches to instruction is to provide some specific examples of its use. Two appropriate examples are the following:

(1) Limits and their precise $\varepsilon-\delta$ definition are replaced by the more intuitive notion of "rounding off" (an idea students have used since elementary school). In other words,

$$\lim_{x \to 0} \frac{x^2 + 2x}{3} = 0$$

is replaced by

$$f(\varepsilon) = \frac{\varepsilon^2 + 2\varepsilon}{3} \sim> 0$$

where if $\varepsilon$ is infinitesimal then $\frac{\varepsilon^2 + 2\varepsilon}{3}$ is very small in size. It is therefore infinitesimal in size, and will round off to zero.

(2) The derivative is not introduced via rotating secants which in the limit become a tangent line at a point on a graph. Rather, the value of the derivative at a point is the slope of the tangent line at that point (if the tangent line exists). This concept of derivative is introduced

after tangent lines (and where they do and do not exist) have been introduced via the intuitive notion of magnification. This is a process whereby a curve is magnified infinitely around a point. If the outcome of magnification looks like a straight line, this line is the tangent line. Both of these examples are based upon the intuitive notion of "close to". In infinitesimal instruction, infinitesimal numbers are used for a more formal, mathematical justification of "close to". Hence, use of the terminology "infinitesimal" in naming this approach to instruction.

The textbook used for this infinitesimal approach to instruction was written specifically for use in calculus courses at Gamma College. According to the author's preface to the text, the text was written with the following aims: to introduce infinitesimals in an intuitive way; to derive the results of calculus using infinitesimal methods with a degree of rigor suitable for a beginning student; and to develop concise, powerful methods of tackling pure and applied problems in analysis. This text is like the text at Alpha University in that the ordering of topics is similar and practice exercises are included at the end of each section. It does however differ substantially in the approach taken for introduction and justification of concepts. This is done exclusively by infinitesimal methods. The topics covered in the course can be found in the extract from the textbook's table of contents found in Appendix K.

The calculus course at Gamma College is structured so that students meet with their instructor for four 50 minute lectures per week. The class was held in a regular classroom at 8:00 a.m. three times per week and 9:00 a.m. for the fourth period. At the beginning of the school term there were about 40 students in Professor Gamma's class.

The researcher noticed that students tended to make lecture notes right in their textbooks, in margins and on the backsides of pages (the text was printed using one side of the page only). Professor Gamma encouraged them to do this because it was conducive to having students not write down notes that were available to them in the text. Instead, they could devote time to involvement in the lecture. The researcher also noticed that students did not stay in the classroom at the end of the lecture to ask Professor Gamma questions. In fact, since other students were waiting outside to come in immediately, Professor Gamma's students were frequently observed following Professor Gamma back to his office after class.

Students in Professor Gamma's class were given lists of textbook exercises to work through in each section of the text. These exercises were not collected for grading for the reason that instructors at Gamma College are responsible for all their own marking. Timewise, along with the fact that introductory calculus students write five term tests, this makes weekly grading of assignments a virtual impossibility. Students at Gamma College write a two hour final exam. Their final grades were determined as follows:

| | |
|---|---|
| Class participation | 5% |
| Best 4 of 5 Chapter Exams | 60% |
| Final Exam | 35% |

Professor Gamma regularly included group problem solving sessions into his lectures. These sessions occurred in 7 of the 23 classes the researcher observed, and varied in length from 10 to 40 minutes. For the problem solving sessions students were divided into groups of 4 to 8. These groups were formed from students sitting near each other that day. Each group was assigned 2 or 3 exercises from the textbook to complete cooperatively. Professor Gamma would collect and read through the solutions, and write comments and suggestions on them.

Professor Gamma incorporated group problem solving sessions into his classes for several reasons. The reasons he outlined at various points during the interview with him included: (1) to encourage self-reliance in learning, (2) to expose students to the ideas of students who are either succeeding in calculus or experiencing similar difficulties, (3) to give students opportunity to talk about difficulties they are experiencing, (4) to provide for mutual support in sorting out difficulties, since "two heads are better than one . . . and they can spur each other on a bit in that regard", and (5) to give students written feedback on their work.

Professor Gamma generally conducted his teaching in a questioning mode. He presented ideas and worked through examples by questioning what was happening and why, what it meant, how one might draw conclusions or proceed to the next step, and why statements or procedures were valid. As a consequence, his instruction explicitly made connections between various aspects of the mathematics, and explicitly justified procedures and conclusions. This is evident in the following extracts from the researcher observation notes:

(September 17)

Ex Rewrite the following spliced function in one line form.

$$y = \begin{cases} x+3 & x < 1 \\ 1 & 1 \leq x < 2 \\ -(x-2)^2 & x \geq 2 \end{cases}$$

*Off and then on . . . light switch . . . you can think of it as a very physical thing. . . . turn it off until you get to the y-axis, then turn it on.*

$$= (x+3)(1 - U(x-1)) \;-\; (x-2)^2\, U(x-2) \;+\; 1(U(x-1) - U(x-2))$$

(pieces of this equation were added throughout the discussion)

*What do I want to do when I'm way over here in the positive region?*
*What do I fill in? What do I leave empty?*

(October 10)

Ex

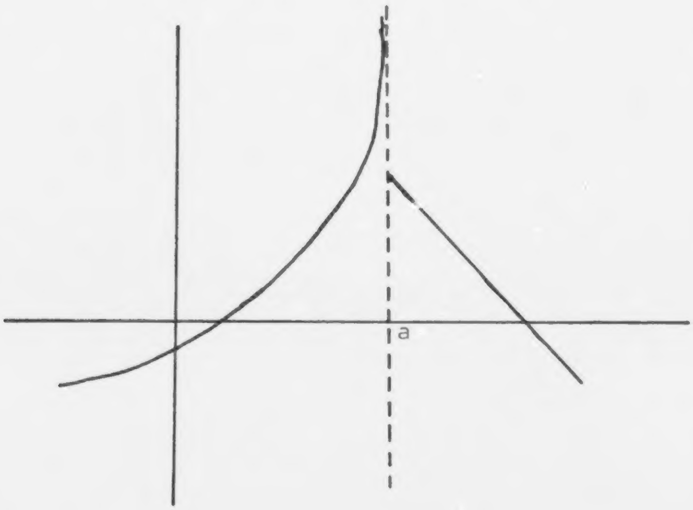


**Figure 13. Graph Drawn by Professor Gamma on October 10**

*As you get close to $a$ what's happening to the values of $y$?*
*- Increasing*
*- Getting bigger*
*How do we designate getting close to $a$ on the x-axis?*
*- $a$ plus $dx$*
*How do you look at corresponding $y$ values?*
- (student mumbles a response)
*$y$ at $a$ plus $dx$. Functional notation.*
*How do we say becomes infinite, mathematically?*

if $y(a+dx) \rightsquigarrow \infty$ (+ or -)
for $dx > 0$ or $dx < 0$.

*Doesn't have to happen on both sides.*

then $x = a$ is called a vertical asymptote.

(November 19)

$x \to a$ is 'similar' to $x = a + dx$ (for any infinitesimal $dx \neq 0$)

*Next is to look at the f(x), the y values*
*How do we look nearby the y-values?*
*- f at a plus dx*

$$\lim_{x \to a} f(x) \text{ is 'similar' to } f(a+dx)$$

*To look in practical terms, what do we have to do with this?*

So round off f(a+dx) is equivalent to the evaluation of the limit.

ie. $\lim_{x \to a} f(x) =$ ___ is equiv. to $f(a+dx) \rightsquigarrow$ ___

*What would it normally round off to?*
*- f at a*
*As long as?*
*- it exists*
*What's that really the same thing as?*
*- continuous*

$$\rightsquigarrow f(a) \text{ (provided } f(x) \text{ is cont. at } x = a)$$

*If it's supposed to be equivalent, then what will this equal?*
*- f at a*
*Again, provided it's continuous. . . . so essentially a definition of continuity using the limit approach.*

$\lim_{x \to a} f(x) = f(a)$ is defn. of continuity at $x = a$ in limit notation.

*Fairly easy to convert just about anything over.*

These three observation extracts demonstrate how Professor Gamma did not write much on the board when presenting ideas. Rather, he focused on talking about the concepts and developing connections amongst symbols, words and graphs. The extracts also show students responding to a number of Professor Gamma's questions. In this class it was generally only a small group of 4 or 5 students who would answer questions. Professor Gamma expressed frustration with this small number because he made effort to structure his presentations to give students opportunity to get involved in and to think about the material. He said:

> . . . typically in class contact with the students you get a lot of them that really don't have the motivation. You get a lot of, like the comment I've mentioned to you before a couple of times. . . . "Why don't you just tell us the answer?" And so I find that kind of frustrating.

In fact, Professor Gamma explicitly stated to his students that a major part of what he wished them to achieve in the course was learning how to think. The course outline given to students at the beginning of the term and discussed in class with them made his intentions clear. It included the following:

> . . . you will be asked many times to think your way through problems you have never seen before. You will even be required to attempt problems which are natural extensions of the material. The purpose of such questions is to see if you are understanding and thinking rather than simply memorizing or falling into a mind set. To be able to attack such problems you need to be fluent with the basics and, more than that, you need to understand the concepts and be able to think logically with them. All of this means you will have to spend time doing calculus. . . . While there is a component of memory work to calculus, it is primarily a course in thinking not memorizing!

In addition to the statements in the above extract, through comments made during lectures, Professor Gamma made clear to students that learning how to think was an objective of the course. Some of his comments were:

> (September 17)
> *I want you to recognize forms. . . . purpose is to make you think about why. Rather than blindly doing things.*
>
> (September 19)
> *I try to teach this course as a how and why. Not just getting the right answers.*

In the interview with Professor Gamma he expanded upon what he saw as the goals of introductory calculus. He saw skills in differentiation and integration, and understandings of the derivative and the integral concepts as goals of the course. However, he emphasized problem solving and learning how to think as the most important aspect. In regards to this he said the following:

> . . . I think the important part of a course like this for students like we have in front of us is their ability to think their way through problems. Their ability to deal with complex issues, and their whole thought process. And that's what's really valuable.

Although Professor Gamma placed value on problem solving skills, he also recognized that many students did not share this perspective. He felt many students' goals were different than what he wished them to achieve in that students wanted pre-determined methods for achieving solutions. Professor Gamma explained this incongruence as follows:

> If they can come out with a better ability to handle a really complex problem. You know, see their way through something without actually knowing how to solve it before they start. That's one of the things I find a real struggle for the students. They want to know how to go about getting an answer before they start solving the problem. They want to know a method that will guarantee them the answer before they start the problem. And what I want them to do is let go of that notion and realize that if they investigate, they can go about investigating a problem and arrive at a solution without knowing some specific method. They can use their basic information and their thought processes to get them through a question to get an answer without having been told the steps they should take to get there. If I could get them to do that I'd be more than happy.

Professor Gamma believed a majority of calculus students are motivated by the fact "they need a good mark so that they can get into this, that, or something else." He also felt there was a broad range of attitudes towards learning amongst his students, and a broad range of willingness to be self-reliant in their own learning. There were those who felt they were "being forced to try to understand something that they don't want to understand". Others enjoyed "the challenge and ability to finally understand a subject instead of just memorizing something." Similarly, there were students who were "reluctant to do things for themselves". Others were at the other extreme in that they were "willing to put in effort" to understand the material. Professor Gamma noted that society puts enormous social and economic pressures on students to achieve good grades. This causes him frustration because it encourages students to concentrate on getting answers. Since he wants them to seek meaningful learning as a goal he tries to foster meaningful learning in the approach he takes to teaching. He explanation of this approach was:

> I find it difficult to know what to say, or how to treat them so they will stop trying to say how do I get an answer? And start trying to say why do I do it this way? What does this really mean? How come? And what does it relate to? And how does it relate to this and that and not something else? I want them to be that way but I find it difficult to know how to do that. And all I'm basically doing at the moment is approaching the course that way myself. I try to teach in such a way where I put those questions in their mind as I teach. And I tend to ask questions on tests that reflect that notion too.

Professor Gamma was in his seventh year of teaching and during that time had taught a variety of undergraduate mathematics courses. What he particularly likes about teaching is the interaction with students, "particularly the one-on-one contact." He finds it "extremely rewarding" when you can help a student "actually understand something." Professor Gamma finds he spends much time outside of class in interactions with students.

He has "an open door policy" which in his words means: "Often I spend the bulk of my day with students in my office."

Professor Gamma was interested in how it is that students actually learn. "The only thing I can put my finger on" he said "is you've got to get down and do the work." He wondered if "the only thing that really gets it together is doing it", "actually doing it and working with the material." He saw these ideas in relation to teaching in the following way:

> But I mean if you teach so that material is clear and reasonably well presented, once you get beyond that I'm not sure there's a lot more you can do. I'm not sure the onus doesn't transfer over to the student, and you say, okay, I'm giving it to you as best I can. Now you've got to take the ball and run with it. And if you don't pick up the ball and start doing something, you're not really going to get it.

Professor Gamma spoke of both visual imagery and language as important aspects of teaching because they make ideas accessible to students. He said he tries to use graphs a lot in his teaching because visual imaging can be extremely important in "latching" onto abstract symbollic representations. According to Professor Gamma this "latching" can occur as follows:

> All you've got is this whole mass of symbols. That you can't put a concrete meaning to easily. You can't put a picture with it that easily. So it's hard to assimilate the information. Where if it's more visual, which calculus tends to be. At least you can do it graphically. There's something else you can latch onto besides the symbols.

With regard to language, Professor Gamma acknowledged that although he generally tries "to put things in terms that are not strictly mathematical", he will "tend to stick closer to the precise terminology" for concepts that are easily misinterpreted. He feels he would be doing students an injustice if he ignored the type of students he had in front of him and adopted the attitude that "to be true to the subject we must develop mathematics in this precise and logical way that has traditionally been done." Thus, Professor Gamma said he tries to say things in a "less sophisticated" way. Although this appears to make students "less afraid of it", it can give rise to difficulties in the following way:

> . . . they also often get misconceptions about what you're saying. Because it's like a parable rather than the actual truth. . . . they draw conclusions that aren't appropriate because of your lack of preciseness.

With regard to using an infinitesimal approach to instruction Professor Gamma felt there were both strengths and weaknesses. He perceived the main strength of the approach to be that "it's a very intuitive way to do things." Even though the notion of hyperreal numbers is seen by some people to be "philosophically objectionable", it can aid in making sense of calculus because it corresponds with calculus notation. The dx's and dy's of calculus notation are a product of the initial way calculus was conceived, with dx and dy corresponding to infinitesimal numbers. Using a limit approach to calculus students are told $\frac{dy}{dx}$ is a symbol, yet the dy and dx are often manipulated as if they were numbers. In comparison, in infinitesimal instruction, dx and dy actually do represent numbers.

According to Professor Gamma, a weakness of an infinitesimal approach to instruction is "the fact that it is different." Students are aware from their friends and previous mathematics courses that other places teach calculus with limits. Some of them therefore "rebel" and ask: "Why are we doing it this way when nobody else does it this way?" Professor Gamma said "anytime you go against the perceived norm there's repercussions from the student level." Another weakness of the course perceived by Professor Gamma is the amount of material that must be covered in one term. It constrains teaching in that "you can't give a concept, and then spend the rest of the week doing examples, and looking at it from different points of view, problem solving with it, and doing all kinds of things with it." If you did that "you'd never get through anything like the amount of material" supposed to be covered.

In terms of their mathematical backgrounds Professor Gamma felt students had "far below what they need as far as preparation in mechanical, algebraic manipulation." He saw this as unfortunate because to do "anything reasonably sophisticated in calculus you've got usually a lot of algebra to deal with." For example, to make use of the definition of the derivative you've usually got to use algebra. Some difficulties might be avoided by "sticking to simple things that don't require a lot of algebra", but if you do that you don't get the entirety and "the meat of the concept". An example Professor Gamma cited was that of continuity. He said many students have the notion that as long as a function is defined it is continuous. He feels this misconception develops from looking at simple functions like polynomials. Only by exposing students to something like a split function, an algebraically more complex entity, can you confront students with their misconceptions.

Professor Gamma believes that for the vast majority of people "math is doable." However, in addition to an incomplete algebraic background, he feels many students are weak in "their ability to use abstract symbols." "They just use them without really having a grasp on them", and since "they're not sure how to put them together" they make up rules. Professor Gamma finds this a problem because having "that much trouble with symbols" is a difficult thing to correct at this stage of a student's mathematics learning.

In summary, introductory calculus at Gamma College uses an infinitesimal approach to instruction. This approach uses language and methods adapted from nonstandard analysis. In particular, an infinitesimal approach to instruction uses rounding off in place of limits, and infinite magnification of a curve around a point in place of a limiting sequence of secant lines at that point. The textbook used at Gamma College was written at the college to support an infinitesimal approach to instruction.

Students at Gamma College are assigned practice exercises that are not collected for grading. Their final grades are determined from five chapter tests and a final exam. Professor Gamma structures his classes so that students are regularly involved in group problem solving sessions. His lectures were generally conducted in a questioning mode, explicitly examining reasons for and connections between various segments of his presentations and various aspects of the related mathematics. In addition, Professor Gamma explicitly emphasized to students that learning to think is a primary objective of introductory calculus. He stated that the development of problem solving abilities is an important aspect of introductory calculus, and sees calculus learning as a matter of doing the work and "doing it" for oneself.

### Class Backgrounds

Data for this section were taken from the Background and End of Term Questionnaires (Appendices A and B). Since the data indicated that characteristics of the three classes were similar, and differed in only a few ways, not all data from the questionnaires is reported. Rather, the data that demonstrates the most relevant similarities or differences with regard to the relationship between class characteristics and potential impacts upon of instruction is reported. The Background Questionnaire provided data on each class at the beginning of the school term. Data from this questionnaire pertaining to students' ages, mathematics and language backgrounds, major subject areas, reasons for attending the post-secondary institution attended, and whether calculus was a required course are reported. The End of Term Questionnaire provided data on each class at the end of the school term. Data from this questionnaire pertaining to students' grades, perceptions of the usefulness of calculus, and exposure to calculus in other courses are reported. The data reported from these two questionnaires is the data that demonstrates the similarities and differences amongst the three calculus classes. Most of the information is reported in percentages, and the remainder is reported as class averages. At Alpha University 88 students completed the Background Questionnaire, and 63 students completed the End of Term Questionnaire. At Beta College 37 students completed the Background Questionnaire, and 25 students completed the End of Term Questionnaire. At Gamma College 43 students completed the Background Questionnaire, and 28 students completed the End of Term Questionnaire.

**Table 1. Selected Portions of the Background Questionnaire Results**

| | Alpha University<br>N = 88 | Beta College<br>N = 37 | Gamma College<br>N = 43 |
|---|---|---|---|
| Average Age: | 19.5 | 19.9 | 19.0 |
| Females: | 50% | 30% | 44% |
| Males: | 50% | 70% | 56% |
| Students who had studied calculus previously: | 26% | 42% | 29% |
| Students who had studied mathematics within the previous 12 months: | 60% | 76% | 72% |
| Students for whom calculus was a required course: | 85% | 81% | 77% |
| Native Language: | | | |
| English | 93% | 76% | 90% |
| Chinese | 2% | 24% | 5% |
| Other | 5% | 0% | 5% |

| | | | |
|---|---|---|---|
| Major Subject Area: | | | |
| Biology | 7% | 5% | 7% |
| Business | 27% | 32% | 14% |
| Chemistry | 6% | 3% | 7% |
| Computer Science | 3% | 22% | 0% |
| Education | 2% | 14% | 9% |
| Mathematics | 2% | 5% | 2% |
| Science | 23% | 0% | 47% |
| Physics | 6% | 0% | 5% |
| Psychology | 7% | 3% | 5% |
| Other | 17% | 16% | 4% |
| Reasons for attending the post-secondary institution attended:* | | | |
| Location | 56% | 35% | 7% |
| Good reputation | 33% | 5% | 16% |
| Institute Size | 0% | 24% | 44% |
| Friends | 0% | 22% | 5% |
| Low grades | 0% | 11% | 9% |
| Christian environment | 0% | 5% | 5% |
| Other | 6% | 16% | 26% |
| No response | 23% | 22% | 19% |

* Column totals are more than 100% because some students gave more than one response to this item.

The data from the Background Questionnaire indicates the three calculus classes similar with respect to students' ages. The average age was between 19 and 20. The data also shows that enrollment in the Alpha University class was more than twice that of the Beta College and Gamma College classes. Data also indicates the Alpha University and Gamma College classes each had about equal numbers of males and females, while the Beta College class had more males than females (70% males). As well, the Beta College class had the highest percentage of students who had previously studied calculus (42% versus 26% and 29%). These differences were not considered meaningful in terms of this study's research questions. However, since language use is a focus of this study, it must be noted that the Beta College class had the highest percentage of students who spoke English as a second language. At Beta College 24% of the students (9 in number) had

Chinese as a native language. One of these 9 students was one of the interview students at Beta College.

More than 60% of the students in each class had studied mathematics in the 12 months previous to the start of this study. For at least 77% of students in each class calculus was a required course. In terms of students' major subject areas, and reasons for attending the post-secondary institution attended, there were some differences between the three classes. Most students at Alpha University were majoring in either a field of science or business (69% in total). A large proportion of students in the Gamma College class also had either business or a science field as a major subject area (81%). In comparison, only 37% of students in the Beta College class had business or science, in particular biology, as a major subject area. Another difference between the three classes were students' reasons for attending the related post-secondary institution. Most Alpha University students named the location of the university as a prime reason for attending the university. In comparison, students at Beta College named location, institute size, or friends as reasons for attending Beta College. At Gamma College, students named location or a good reputation as reasons for attending Gamma College.

**Table 2. Selected Portions of the End of Term Questionnaire Results**

| | Alpha University<br>N = 63 | Beta College<br>N = 25 | Gamma College<br>N = 43 |
|---|---|---|---|
| Average course grade before the final exam (as reported by the students): | 68% | 63% | 60% |
| Students receiving a lower grade than they had been expecting: | 38% | 72% | 71% |
| Students who saw calculus as useful to their future career: | 44% | 44% | 39% |
| Students who saw calculus as useful to society: | 63% | 48% | 61% |

Data from the End of Term Questionnaire indicates students from the three classes were similar with regards to their calculus grades and views of the usefulness of calculus. In addition, the data indicates class grades at the end of the school term were similar amongst students at the three institutions. The three class averages for these grades were between 60% and 68%. However, a higher percentage of students at Beta College and Gamma College as compared to Alpha University were receiving lower calculus grades than they had expected (72% and 71% respectively). Only 38% of the Alpha University students were receiving lower calculus grades than they had expected.

In summary, the three calculus classes were different in size, with enrollment at Alpha University more than double that of Beta College and Gamma College. The other main difference between the classes was a higher percentage of students in the Beta College class for whom English was a second language. None of the differences at either the beginning or end of the school term between the three classes were considered significant to this study's research questions.

### Classroom Observations

Classroom observations were done to provide a description of instruction in each of the three calculus classes. These descriptions addressed the fourth and fifth research objectives related to the nature of the three approaches to instruction as delivered to students and the potential impact of each instructional approach on student learning. The information gathered was used to qualitatively describe each instructional setting (Section A of this chapter). The information was also used to describe the following for each class: (1) relative time spent on concept development and use of examples, (2) *language use*, and (3) *sources of conviction*. In this section results of the systemic classroom observations are first reported. The similarities and differences between instruction in the three classes are then discussed.

At each institution 25% to 50% of regular class periods were observed. Not all classroom observation sessions were coded on Classroom Observation Summary Sheets because several classes at each institution were observed at the start of the school term to practice note-taking procedures and refine observation coding categories (see Appendices C and D). Classes at Alpha University were observed on 20 occasions and notes from these observations were coded on 16 occasions. As well, labs at Alpha University were observed a total of 6 times, with 5 observations coded. Classes at Beta College were observed 14 times and 11 of these observations were coded. At Gamma College, classes were observed a total of 22 times. Observation notes were coded on 15 of these occasions. Group problem solving sessions occurred in 6 of the 7 remaining observations. Coding was not done for classroom observations when group problem solving sessions occurred because these sessions generally used 40% to 80% of class time.

After the school term was completed data from the Classroom Observation Summary Sheets were used for further instructional analysis. Information from one classroom observation session (25 two minute intervals) was summarized on a Classroom Data Analysis Sheet. This sheet can be found in Appendix L and a sample of a completed sheet is in Appendix M. The rows of the Classroom Data Analysis Sheet correspond to the variable categories of the Classroom Observation Summary Sheet. As with the Classroom Observation Summary Sheet, a distinction between written and spoken language is made on the Classroom Data Analysis Sheet. However, the same distinction for *sources of conviction* was not maintained, and no distinction was made between Context codes entered in the Written Language column and those entered in the Spoken Language column (see Appendices C and D). The main reason for these modifications was that the distinctions did not appropriately describe features of instruction, because these features were usually integrated. For example, the spoken *sources of conviction* displayed in instruction were generally straightforward verbalization or reading of what was written on the board.

Each column of the Classroom Data Analysis Sheet was completed by entering circles in the rows whose codes appeared in the corresponding two minute time interval (row) of the Classroom Observation Summary Sheet. Thus, each classroom observation was summarized on a single sheet. The last column of the sheet was used for recording the total number of entries (circles) in each row. When the total of each row was summed across all sheets for an institution, total numbers were obtained for each institution. These values are found in Table 3. The totals represent the total number of two minute time intervals observed that displayed the observation code variable of that row. The percentage of two minute time intervals observed that displayed the observation code variable of the related row are given in parentheses in Table 3. The values of N for the columns of this table are the total number of two minute intervals coded for the related set of observations. Explanations of the category codes of the first column are found in Chapter 3 and also in Appendix E.

**Table 3 - Total Numbers for Each Institution for Each Classroom Observation Analysis Code**

| | Institution | | | |
|---|---|---|---|---|
| **Row** | **Alpha University N=400** | **Beta College N=275** | **Gamma College N=350** | **Alpha University Labs N=75** |
| CP | 148.5 (37%) | 112 (41%) | 132.5 (38%) | 2 (3%) |
| EX | 190 (48%) | 150 (55%) | 211 (60%) | 64.5 (86%) |
| O | 63 (16%) | 23 (8%) | 33 (9%) | 58.5 (78%) |
| MC | 254 (64%) | 170.5 (62%) | 286 (82%) | 45 (60%) |
| PC | 86.5 (22%) | 112 (41%) | 107.5 (31%) | 10 (13%) |
| CF | 45.5 (11%) | 16.5 (6%) | 6 (2%) | 21 (28%) |
| TL(w) | 284.5 (71%) | 206.5 (75%) | 257 (73%) | 60.5 (81%) |
| TL(s) | 44 (11%) | 25 (9%) | 17 (5%) | 3.5 (5%) |
| EL(w) | 14.5 (4%) | 13.5 (5%) | 7 (2%) | 4 (5%) |
| EL(s) | 133 (33%) | 132.5 (48%) | 216 (62%) | 9 (12%) |
| M | 186.5 (47%) | 155.5 (57%) | 273.5 (78%) | 21.5 (29%) |
| IE | 65 (16%) | 89 (32%) | 96.5 (28%) | 2 (3%) |
| ER | 94.5 (23%) | 37.5 (14%) | 16 (5%) | 18 (24%) |
| EO | 33 (8%) | 15.5 (6%) | 16 (5%) | 28.5 (38%) |

Values in Table 3 indicate instruction at the three institutions was similar in the percentage of two minute intervals in which the following occurred: (1) concept presentation (CP), (2) presentation of examples (EX), (3) a context free presentation (CF), (4) use of written or spoken *technical language* (TL(w) or TL(s)), (5) use of written *everyday language* (EL(w)), and (6) use of rules and other external *sources of conviction* (ER and EO combined).

Values in Table 3 indicate differences in instruction occurred at the three institutions in the following ways with respect to the percentage of two minute intervals in which particular events occurred:

(1) Gamma College instruction displayed a higher percentage of mathematical contexts (MC) than Alpha University or Beta College instruction (82% versus 64% and 62%, respectively). This fact means that infinitesimal instruction as implemented by Professor Gamma more frequently developed mathematical contexts within which ideas and examples were presented.

(2) Beta College instruction displayed a slightly higher percentage of physical contexts (PC) than Alpha University or Gamma College instruction (41% versus 22% and 31%, respectively). This fact means concepts-first instruction as implemented by Professor Beta more frequently incorporated graphs and other physically oriented calculus interpretations and justifications.

(3) Gamma College instruction displayed a higher percentage of use of spoken everyday language (EL(s)) than Alpha University or Beta College instruction (63% versus 33% and 48%, respectively). That is, infinitesimal instruction as implemented by Professor Gamma incorporated more extensive use of *everyday language* interpretations of calculus ideas and procedures.

(4) Gamma College instruction displayed a higher percentage of use of mathematics as a source of conviction (IM) than did Alpha University or Beta College instruction (78% versus 47% and 57%). This fact means infinitesimal instruction as implemented by Professor Gamma more frequently explicitly justified statements or procedures by reference to previously established mathematics statements or procedures.

The implications of these findings in relation to the research issues of this study are that if instruction influences students' *language use* and *sources of conviction*, then students at Gamma College would be expected to use more *everyday language* than other students. They would also be expected to display more use of mathematics as a *source of conviction*. Further, since classroom instruction at Beta College displayed more use of physical contexts (usually graphs), it would be expected that Beta College students would display more use of physical contexts in their problem responses. These potential impacts upon student learning are important in that if they occur they indicate instruction can affect students' *language use* and *sources of conviction*.

## Textbook and Exercise Assignment Analysis

Textbook and exercise assignment analysis was done to provide descriptions of the written instructional materials of the three calculus courses. These descriptions provide information on the ways each instructional approach as reflected in the related textbook and exercise assignment was translated into instructional events. The analysis was structured to provide descriptions in terms of relative time spent on concept development and use of examples, *language use*, and *sources of conviction*. As already outlined in Chapter 3, analysis categories for the textbooks and exercise assignments were designed to allow as much correspondence as possible to the classroom observation analysis categories. The Textbook Analysis Summary Sheet was developed for this purpose (Appendix N). Entries were made in the Type column on this sheet only when an Example (EX) or Exercise (EXC) code was entered in the Event column. Thus, entries made in the Type column naturally fell into two disjoint classifications: Examples or Exercises. The distinctions in these classes between Imitation and Non-Imitation Examples, and between Routine and Non-Routine Exercises were developed as explained in Chapter 3. The various subcategories, as well as the "Transitory" category emerged from inductive analysis of the textbook examples and exercise assignments. Details of the categories and subcategories are discussed below. Summary descriptions and codes are in Appendix O.

### Examples

The following question was asked of each textbook example: "Could students use this example to learn by imitation?" That is, could students duplicate the steps followed in the example to work through a variety of exercise questions similar to the example? The answer to this question determined the classification of an example as Imitation (I) or Non-imitation (N). Placement in one of the subcategories was made subsequently. In addition, if a visual component such as a graph or diagram was included in the example, then a (v) was included at the end of the category code entered under the Type column. The Example categories and subcategories are defined below. The code for each is given in brackets next to the name for that category or subcategory.

**A. (I) Imitation**

Students could duplicate the steps in the example with a variety of exercise questions similar to the example.

(1) **(d) demonstration**: demonstration of a type of calculation or procedure, or application of a rule.

(2) **(p) property**: a specified property is displayed through a graph, equation, or numerical or algebraic expression.

(3) **(w) word problem**: a one or two step application of a concept or procedure. This application is to a physical context, as opposed to application to another area of mathematics.

**B. (N) Non-Imitation**

Students are not likely to be able to duplicate the steps in the example with a variety of exercises similar to the example.

(1) **(m) multistep**: an application (within either a mathematical or physical context) of concepts or procedures that involve one or more of the following in reaching a solution: analysis of a situation, synthesis of several concepts, or construction of a graph, equation or expression.

(2) **(i) interpretation**: interpretation or explanation of a graph or mathematical or physical situation.

**(v) Visual Component** : a graph or diagram is present in the example.

Textbook example codes were entered under the Type column with the category code (capital letter) followed by the subcategory code (lower case letter). As well, (v) was included at the end if the example included a visual component. The possible entries were therefore: Id, Id(v), Ip, Ip(v), Iw, Iw(v), Nm, Nm(v), Ni, and Ni(v). Examples of textbook examples for each of the five subcategories are given below. The code assigned to each example is given in brackets at the start of the example, as is the textbook it was taken from. For confidentiality reasons the textbooks at Beta and Gamma Colleges are not explicitly named (see Appendix V).

## A. (I) Imitation

### (1) Subcategory: (d) demonstration

(Stewart, 1987; p.62) (Code: Id)
**Example** Find $\lim_{t \to 2} \frac{\sqrt{t} - 2}{t - 4}$

(Beta College textbook) (Code: Id)
**Example** Demonstrate the validity of the quotient rule for

$$y = \frac{x^5 + 2x^3}{x^2}$$

(Gamma College textbook) (Code: Id(v))

**Example** Find the tangent line to $y = x^2 - x$ at $(0,0)$ [a graph is drawn].

### (2) Subcategory: (p) property

(Stewart, 1987; p.30) (Code: Ip(v))
**Example** Sketch the graph of $f(x) = x^2$ [a graph is drawn].

(Gamma College textbook) (Code: Ip(v))
**Example**
**DNE (does not exist)** This is the case if the curve on the infinitely magnified diagram is not a straight line. From the definition of the derivative we can identify cases where the derivative does not exist [five different graphs which have points where a derivative does not exist are then given].

### (3) Subcategory: (w) word problem

(Stewart, 1987; p.143) (Code: Iw)
**Example** The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, where $s$ is measured in centimeters and $t$ in seconds. Find the acceleration as a function of time. What is the acceleration after 2 s?

2. (Beta College textbook) (Code: (Code: Iw(v))
**Example** As a spherical weather balloon rises its radius increases at 0.5 cm/minute. How fast is its volume changing when its radius is 20 cm? [ a diagram is included]

(Gamma College textbook) (Code: Id(v))
**Example** The number of rabbits in a hutch is given by $R = 4t + 2$, $t$ in months. Find the rate of change of $R$ with respect to $t$? [a figure of rabbits in a hutch is included]

## B. (N) Non-Imitation

### (1) Subcategory: (m) multistep

(Stewart, 1987; p.39) (Code: Nm)
**Example** Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x+3$

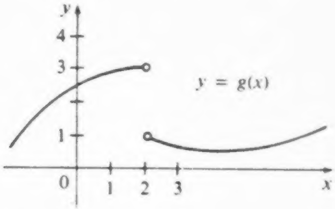
(Beta College textbook) (Code: Nm)
**Example** If $f(x) = \frac{1}{x}$ determine $f^{(n)}(x)$ and evaluate $f^{(8)}(x)$.

(Gamma College textbook) (Code: Nm(v))
**Example** Find the right circular cylinder of maximum volume that can be inscribed in a sphere of radius $R$ [a diagram is included].

**(2) Subcategory: (i) interpretation**

(Stewart, 1987; p.66) (Code: Ni(v))

**Example**



State the values (if they exist) of

$$\lim_{x\to 2^-} g(x) \qquad \lim_{x\to 2^+} g(x) \qquad \lim_{x\to 2} g(x)$$

**Figure 14. Alpha University Textbook Example for the Interpretation Example Subcategory**

(Beta College textbook) (Code: Ni(v))

**Example** Consider $f(x) = 2x^3 + 3$. We ask: Is there a number that $f(x)$ gets arbitrarily close to when we let $x$ get as close to 1 as we like, subject to the condition $x \neq 1$? [a graph of $f(x)$ is included]

(Gamma College textbook) (Code: Ni(v))

**Example**

$F'(-\infty) = 0^+$
$F'(-1) = \text{DNE}$
$F'_+(-1) = -1$
$F'(-1^+) = -1$
$F'(0) = -1$
$F'(1) = \text{DNE}$
$F'_-(1) = +\infty$
$F'_+(1) = -1.5$
$F'(3) = \text{DNE}$
$F'_+(3) = \text{DNE}$
$F'(3^-) = -\infty$
$F'(3^+) = +\infty$
$F'(5) = 0$
$F'(+\infty) = 0$

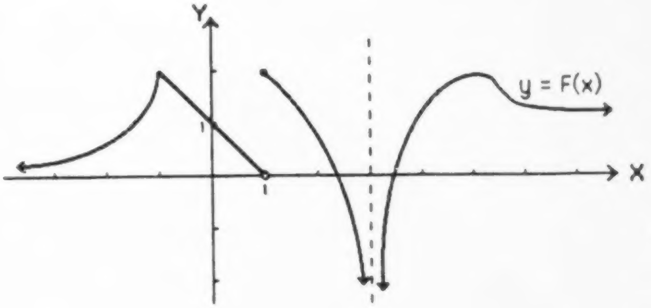


**Figure 15. Gamma College Textbook Example for the Interpretation Example Subcategory**

In relation to the example categorization scheme it should be noted that, according to constructivism, what a student does with a particular example is an individual matter. One student might make use of a particular example by trying to imitate its steps with other exercises, while another student might make use of the example in terms of trying to understand its overall structure, reasons for specific steps, and connections to related concepts. Further, any example might at some point be used to learn by imitation. This imitation could occur if a student is given an exercise to complete which is almost identical to the example. For example, the solution to a multistep problem such as the second multistep example given above (nth derivative of $\frac{1}{x}$) could be imitated if a student were asked to find the nth derivative of the function $y = \frac{1}{x^2}$. However, the designation of an example as Imitation or Non-Imitation was made according to whether steps could be imitated with a large variety of exercise questions, rather than a particular exercise.

Total numbers for each example category and subcategory for each textbook are in Table 4. The percentage of examples in each textbook within each category and subcategory are also in Table 4 in parentheses.

**Table 4 - Total Numbers and Percentages of Textbook Example Categories and Subcategories**

| | Category Code | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Textbook** | **Id** | **Id(v)** | **Ip** | **Ip(v)** | **Iw** | **Iw(v)** | **Total I** | **Nm** | **Nm(v)** | **Nl** | **Nl(v)** | **Total N** | **Total (v)** | **Total** |
| Alpha University (Stewart, 1987) | 89<br>(40%) | 58<br>(26%) | 7<br>(3%) | 8<br>(4%) | 4<br>(2%) | 1<br>(.5%) | 165<br>(74%) | 27<br>(12%) | 18<br>(8%) | 1<br>(.5%) | 11<br>(5%) | 57<br>(26%) | 96<br>(43%) | 222 |
| Alpha University Lab Manual | 64<br>(57%) | 33<br>(29%) | 0 | 0 | 0 | 0 | 97<br>(86%) | 4<br>(4%) | 12<br>(11%) | 0 | 0 | 16<br>(14%) | 45<br>(40%) | 113 |
| Beta College | 82<br>(51%) | 34<br>(21%) | 0 | 1<br>(.6%) | 5<br>(3%) | 0 | 122<br>(76%) | 18<br>(11%) | 6<br>(4%) | 1<br>(.6%) | 13<br>(8%) | 38<br>(24%) | 51<br>(32%) | 160 |
| Gamma College | 39<br>(15%) | 145<br>(56%) | 32<br>(12%) | 5<br>(2%) | 11<br>(4%) | 4<br>(2%) | 236<br>(91%) | 5<br>(2%) | 4<br>(2%) | 0 | 13<br>(5%) | 22<br>(9%) | 171<br>(66%) | 258 |

The values in Table 4 show that all three textbooks and the Alpha University lab manual were similar in the distribution between Imitation (I) and Non-Imitation (N) examples. At least 74% of the examples in each book were classified as Imitation, while at most 26% were classified as Non-Imitation (N). Within the example subcategories the textbooks were also similar. Most examples classified as Imitation (I) fell into the demonstration (d) subcategory. Few examples fell into the property (p) and word problem (w) subcategories. The one difference that stands out in the Imitation (I) subcategories is the distribution in the demonstration (d) subcategory between examples that include a visual component (v) and those that do not. In the Gamma College textbook over 50% of the demonstration examples (d) contained a visual component, while only 15% did not include a visual component. The use of a visual component within a demonstration example was lower for the other textbooks. That is, they contained a higher percentage of demonstration examples which did not include a visual component. The higher percentage of examples in the Gamma College textbook containing a visual component (v) is also reflected in the values in the Total (v) column of Table 4.

It is noteworthy that the textbooks for the three different approaches to instruction are similar in the nature of their examples. Although the Alpha University and Gamma College textbooks contained more examples, the type of examples they used were similar in nature to the examples in the Beta College textbook. Thus, it can be stated that the examples used in written instructional materials for each of the three institutions did not reflect different instructional events amongst the three classes.

### Exercises

To attempt to classify exercises as Routine or Non-Routine the following question was asked of each exercise: "Could students do this exercise by recall, or by simply following rules or procedures?" As explained in Chapter 3, this question could not be answered for all exercises. In addition, not all the subcategories of Christiansen & Walther's (1986) two-column categorization were appropriate for the calculus textbooks being examined. First of all, from examination of these textbooks there emerged a set of exercises which could not be classified as Routine because they involved more than predetermined application of rules. Furthermore, these exercises could not be classified as Non-Routine because procedures leading to a solution had been presented in previous sections of the textbook. Thus, a middle category for "Transitory" exercises emerged.

Further examination of exercises revealed Christiansen & Walther's subcategories for Routine Exercises (recognition, algorithm, and word problem exercises) to be appropriate for classifying Routine Exercises. However, the subcategories they outlined for Non-Routine Exercises (process, open search, and problem situation exercises) were not appropriate for classifying exercises in the calculus textbooks. From examination of the features of the textbook exercises there emerged a different set of Non-Routine Exercise subcategories, which were named "Problem" Exercise subcategories. Similarly, there emerged two subcategories for exercises classified as Transitory.

Examination to develop the subcategories was done by rotating through sections of the textbooks. That is, examination proceeded in the following order: a section from the Alpha University textbook or lab manual, a section from the Beta College textbook, then a section from the Gamma College textbook. For each exercise, a short phrase describing the nature of a solution to the exercise was written down. The descriptions were then grouped into the subcategories described in the outline given bellow. Category and subcategory codes in this outline are given in brackets next to their names.

**A. (R) Routine**

Tasks for which a procedure leading to a solution has been presented in the textbook.

(1) **(i) identification**: identification or recognition of a property or concept.

(2) **(a) algorithm**: use of a rule or algorithm.

(3) **(w) word problem**: a one or two step application of a concept or procedure. This application is to a physical context, as opposed to application to another area of mathematics.

**B. (T) Transitory**

Tasks for which procedures leading to a solution have been presented in the textbook, but the solution procedures involve several steps, or interpretation of notation or graphs.

(1) **(g) graphing**: application of rules along with graphing of the results.

(2) **(a+) application**: use of several rules or algorithms, or use of a rule or algorithm that involves interpretation of notation or interpretation of a graph.

## C. Problem (P)

Tasks for which a procedure leading to a solution is not known.

(1) **(m) multistep**: a task involving more than one of the following: identification of a property or concept, analysis of a situation, synthesis of concepts or calculation results, application of rules or algorithms, derivation of an equation or formula, or sketching of a graph.

(2) **(c) create**: create an example of a situation, function or equation that possesses specified properties (graphs are not included here).

(3) **(cg) construct a graph**: construct a graph which possesses specified properties.

(4) **(p) prove**: prove a general result.

(5) **(e) explain**: explain, describe or interpret a mathematical situation (graphs are not included here).

(6) **(ig) interpret a graph**: interpretation of a graph.

Exercise codes were entered under the Type column of the Textbook Analysis Summary Sheet, with the category code (capitol letter) followed by the subcategory code (lower case letter). The possible entries were therefore: Ri, Ra, Rw, Tg, Ta+, Pm, Pc, Pcg, Pp, Pe, and Pig. Examples of exercises for each of these eleven subcategories are given below. The code assigned to each example is given in brackets at the start of the example, as is the textbook it was taken from. As with the textbook examples, for reasons of confidentiality the textbooks at Beta and Gamma Colleges are not explicitly named (see Appendix V).

## A. Routine (R)

### (1) Subcategory: (i) identification

(Gamma College textbook) (Code: Ri)
State whether the given equation is a polynomial, rational, algebraic, or transcendental function.

a. $y = x^3 + \sqrt{x} + 1$

b. $y = \frac{1}{x} + 5$

c. $y = 3x^5$

d. $y = 3^{5x}$

e. $y = \begin{cases} 2, & x \neq 0 \\ x, & x = 0 \end{cases}$

f. $y = 17$

### (2) Subcategory: (a) algorithm

(Stewart, 1987; p.139) (Code: Ra)
Find dy/dx by implicit differentiation.

$$y^5 + 3x^2y^2 + 5x^4 = 12$$

(Beta College textbook) (Code: Ra)
Determine $s'(8)$ if $s(t) = \dfrac{3t + 6}{\sqrt[3]{t}}$

(Gamma College textbook) (Code: Ra)
Use the definition of the derivative to find the derivative.

$$y = \frac{1}{x^2 + 3}$$

### (3) Subcategory: (w) word problem

(Beta College textbook) (Code: Rw)
The distance, in meters, that an object falls in t seconds when dropped from the edge of a cliff is given by $h(t) = 4.9t^2$. Determine the average velocity of the object in each of the following intervals and make a conjecture about the instantaneous velocity when $t = 2$.
(a) $t = 2$ seconds to $t = 2.1$ seconds.
(b) $t = 2$ seconds to $t = 2.01$ seconds.
(c) $t = 2$ seconds to $t = 2.001$ seconds.

(Gamma College textbook) (Code: Rw)
Big Al's weight at a distance r miles from the earth's center is given by

$$W = \frac{5 \times 10^9}{r^2}$$

Find his weight at the earth's surface, $r = 4000$ miles. What is his rate of change of weight with respect to height there? Interpret.

## B. Transitory (T)

### (1) Subcategory: graphing (g)

(Stewart, 1987; p.199) (Code: Tg)
Find (a) the intervals of increase or decrease, (b) the local maximum and minimum values, (c) the intervals of concavity, and (d) the x-coordinates of the points of inflection. Then use this information to sketch the graph.

$y = x^4 - 6x^2$

(Beta College textbook) (Code: Tg)
Use the first derivative test to determine the local maxima and minima and to sketch the graphs of the following functions.

(a) $y = x^2 - 6x + 20$
(b) $f(x) = x^4 - 4x^3 - 8x^2 + 3$

(Gamma College textbook) (Code: Tg)
Find the tangent line to $y = \dfrac{5}{1 + x^2}$ at $x = 2$. Graph.

### (2) Subcategory: (a+) application

(Stewart, 1987; p.164) (Code: Ta+)
Use differentials to find an approximate value for the given number.

(a) $\sqrt{99}$ (b) $(1.97)^6$

(Beta College textbook) (Code: Ta+)
Determine $\dfrac{d^2y}{dx^2}$ if $16x^2 + 25y^2 = 400$.

(Gamma College textbook) (Code: Ta+)
Use the Intermediate Value Theorem to prove that $y = 5x^7 + 3x - 7$ has a zero on the interval $0 \le x \le 1$.

## C. Problem (P)

### (1) Subcategory: (m) multistep

(Stewart, 1987; p.187) (Code: Pm)
Suppose F is continuous on [2,5] and $1 \le f'(x) \le 4$ for all x in (2,5). Show that $3 \le f(5) - f(2) \le 12$.

(Beta College textbook) (Code: Pm)
A mass of 100 kg is to be raised by a lever with the fulcrum at one end and the applied force at the other. If the 10 kg mass is to be 1 m from the fulcrum and the lever has a mass of 4 kg/m what length of lever is required so that the applied force can be as small as possible?

(Gamma College textbook) (Code: Pm)
Find the points P on the curve $y = x^2$ where the tangent line at P has x-intercept 4.

**(2) Subcategory: (c) create**

(Beta College textbook) (Code: Pc)
Find a third degree polynomial $P(x)$ such that $P(1) = 1$, $P'(1) = 3$, $P''(1) = 6$, and $P^{(3)}(1) = 12$. Hint: Let $P(x) = ax^3 + bx^2 + cx + d$.

(Gamma College textbook) (Code: Pc)
Show that in general

(a) $\frac{d^2y}{dx^2} \neq (\frac{dy}{dx})^2$

(b) Find a function for which $d^2y/dx^2 = (dy/dx)^2$

**(3) Subcategory: (cg) construct a graph**

(Beta College textbook) (Code: Pcg)
Show by exhibiting the graph of an example that if $y = f(x)$ is increasing for every x in an interval (a,b) then $y' = f'(x)$ may be a decreasing function in the interval (a,b).

(Gamma College textbook) (Code: Pcg)
For the curve $y = x^2 + x$, find and graph the following magnifications about (0,0).
a. $m = 1$
b. $m = 10$
c. $m = 100$
d. $m = I_0$

**(4) Subcategory: (p) prove**

(Stewart, 1987; p.140) (Code: Pp)
Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at the point $(x_0,y_0)$ is

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

(Beta College textbook) (Code: Pp)
Use the product rule twice to demonstrate that if f, g, and k are differentiable functions, then

$$(fgk)' = f'gk + fg'k + fgk'$$

(Gamma College textbook) (Code: Pp)
Prove that if $\varepsilon$ is a positive infinitesimal, then $1/\varepsilon$ is a positive infinite number.

### (5) Subcategory: (e) explain

(Beta College textbook) (Code: Pe)
What do the following limits imply about H-asymptotes?

$$\lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to \infty} g(x) = 0$$

(Gamma College textbook) (Code: Pe)
Can you determine the velocity of an object
a. from a single high speed photograph?
b. from two single high speed photographs?
c. from a single slow speed photograph?
Explain.

### (6) Subcategory: (ig) interpret

(Beta College textbook) (Code: Pig)
Identify each point on the graph below where the derivative does not exist and explain why.

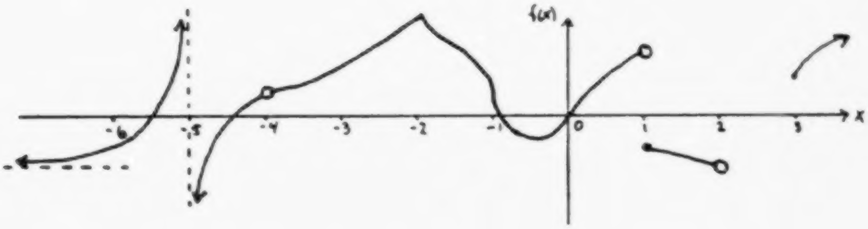


Figure 16. Beta College Exercise Example for the Interpret Exercise Subcategory

(Gamma College textbook) (Code: Pig)
Use graphical differentiation (slope interpretation of derivative) to sketch the graph of dy/dx for each of the following.

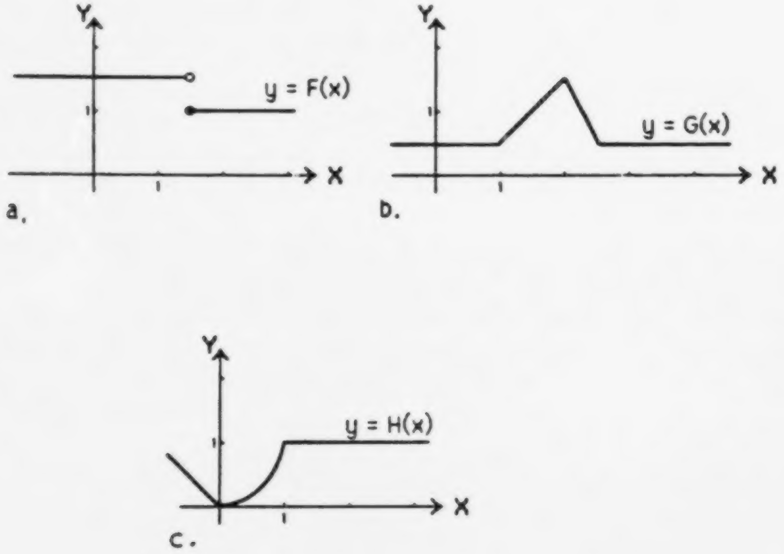


**Figure 17. Gamma College Exercise Example for the Interpret Exercise Subcategory**

Total numbers for each exercise category and subcategory for the various textbooks and exercise assignments are in Table 5. The exercises included in analysis of the Alpha University textbook (Stewart, 1987) are those assigned to students in their weekly graded assignments. For the lab manual at Alpha University, all exercises were analyzed because no specific subset of these exercises was assigned to students. Exercises for the Beta College course were analyzed as three distinct sets: (1) exercises found within the body of a textbook section, (2) exercises found at the end of a textbook section, and (3) weekly assignment exercises given to students on separate sheets. As with the Alpha University lab manual, all exercises at the end of a section of the Beta College textbook were analyzed because no specific subset of these exercises was assigned to students. The exercises analyzed from the Gamma College textbook were those on a "recommendation" list given to students.

# Table 5 - Total Numbers and Percentages of Textbook and Assignments Exercise Categories and Subcategories

| | Code | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Source** | Rl | Ra | Rw | Total R | Tg | $Ta^+$ | Total T | Pm | Pc | Pcg | Pp | Pe | Plg | Total P | **Total** |
| **Alpha University Assignments** | 0 | 71<br>(60%) | 1<br>(1%) | 72<br>(61%) | 20<br>(17%) | 9<br>(8%) | 29<br>(25%) | 15<br>(13%) | 0 | 0 | 2<br>(2%) | 0 | 0 | 17<br>(15%) | 118 |
| **Alpha University Lab Manual** | 0 | 133<br>(75%) | 0 | 133<br>(75%) | 30<br>(17%) | 0 | 30<br>(17%) | 5<br>(3%) | 0 | 0 | 0 | 0 | 9<br>(5%) | 14<br>(8%) | 177 |
| **Beta College Embedded Exercises** | 0 | 43<br>(55%) | 2<br>(3%) | 45<br>(58%) | 11<br>(14%) | 9<br>(12%) | 20<br>(26%) | 4<br>(5%) | 0 | 0 | 7<br>(9%) | 1<br>(1%) | 1<br>(1%) | 13<br>(16%) | 78 |
| **Beta College End of Section Exercises** | 0 | 119<br>(51%) | 0 | 119<br>(51%) | 25<br>(11%) | 15<br>(6%) | 40<br>(17%) | 50<br>(21%) | 2<br>(1%) | 8<br>(3%) | 2<br>(1%) | 0 | 14<br>(6%) | 76<br>(32%) | 235 |
| **Beta College Weekly Assignments** | 0 | 77<br>(52%) | 2<br>(1%) | 79<br>(53%) | 9<br>(6%) | 14<br>(9%) | 23<br>(15%) | 14<br>(9%) | 1<br>(1%) | 7<br>(5%) | 2<br>(1%) | 1<br>(1%) | 23<br>(15%) | 48<br>(32%) | 150 |
| **Gamma College Recommended Exercises** | 19<br>(3%) | 276<br>(41%) | 16<br>(2%) | 311<br>(46%) | 90<br>(13%) | 75<br>(11%) | 165<br>(24%) | 60<br>(9%) | 18<br>(3%) | 6<br>(1%) | 31<br>(5%) | 36<br>(5%) | 52<br>(8%) | 203<br>(30%) | 678 |

Limited conclusions only can be made with the values in Table 5 because of the nature of the exercise sets. For example, all exercises at the end of sections of the Alpha University lab manual and Beta College textbook were examined, but it is not known which of these exercises students actually attempted. It is also not known which exercises embedded in the body of a section of the Beta College textbook students were likely to have attempted. In addition, although specific exercises were assigned to students from the Alpha University and Gamma College textbooks, it is not known which additional exercises students might have attempted. It is likely that students at Gamma College attempted additional exercises because exercises assigned in group problem solving sessions frequently included questions additional to those on the "recommendation" list. As well, students at all three institutions had available to them test and exam questions from previous school terms.

The conclusions that can be made from Table 5 are that the Gamma College textbook contained certain types of exercises that were not present to the same extent in the other two textbooks. In particular, the following differences can be seen:

(1) The Gamma College textbook contained a higher number of Routine (R), Transitory (T), and Problem (P) exercises. This finding means students at Gamma College were required to complete more exercises than students at the other two institutions (although whether they did or not is not actually known).

(2) The Gamma College textbook contained exercises classified as Routine-identification (Ri) exercises, but the other books did not contain such exercises. This fact means that, in addition to Routine exercises requiring use of an algorithm (Ra) or solution of a one or two step word problem (Rw), Gamma College students were asked to do exercises involving identification or recognition of a property or concept (Ri).

(3) The Gamma College textbook contained exercises classified as Problem-create (Pc), Problem-prove (Pp), and Problem-explain (Pe). This finding means students at Gamma College, more so than the other two groups of students, were required to create an example of a situation, function or equation possessing specific properties, to prove a general result, or to explain, describe or interpret a mathematical situation.

In relation to the research objectives of this study these findings indicate the exercises of the instructional materials for the three approaches to instruction did not always contain similar instructional exercise events. More specifically, the infinitesimal approach to instruction as reflected in the instructional materials for Gamma College differed from the other two approaches to instruction in the nature of the exercises given to students.

To complete the instructional materials analysis all the data on the Textbook Analysis Summary Sheets (Appendix N), including the example (EX) and exercise (EXC) events, were used. A completed sample of a Textbook Analysis Summary Sheet is in Appendix P. For copyright and confidentiality reasons the corresponding section of the textbook is not given. Once the Textbook Analysis Summary Sheets were completed, further analysis of the information on these sheets was carried out in a similar way to analysis of the corresponding information on the Classroom Observation Summary Sheets. The analysis used information entered in the Event, Language and Convictions columns of the Textbook Summary Sheets. Information from the Type column was analyzed separately, as already described.

The Textbook Data Analysis Sheet (Appendix Q) was developed to complete analysis of the information on the Textbook Analysis Summary Sheets. It is almost identical to the Classroom Data Analysis Sheet. The differences are: (1) addition of a row corresponding to the Event category "Exercise" (EXC), (2) removal of the rows for spoken technical or everyday language, and (3) removal of the row for an External-Other (EO) source of conviction. Another difference is that the 25 columns are not "equal" in that they do not correspond to equal amounts of textbook material. Rather, each column corresponds to an Event (concept presentation, example, or exercise). Depending on the number of events in a particular textbook section or exercise assignment, part of a sheet or two sheets were required to record information from the Textbook Analysis Summary Sheet for that section.

The Textbook Data Analysis Sheets were completed in the same way as were the Classroom Data Analysis Sheets. For each column (Event), circles were entered in the rows whose codes appeared in the corresponding row of the Textbook Analysis Summary Sheet. The final column was used for recording the total number of entries (circles) in each row. When the total of each row was summed across all sheets for a textbook or exercise assignment, total numbers and related percentages for each code were obtained. These values are in Table 6.

Table 6 - Total Textbook Numbers and Percentages for Each Institution for Each Textbook Analysis Code

| | Textbook | | | |
|---|---|---|---|---|
| **Row** | **Alpha University N = 341** | **Beta College N = 275** | **Gamma College N = 426** | **Alpha University Lab Manual N = 161** |
| CP | 129 (38%) | 117 (43%) | 168 (39%) | 48 (30%) |
| EX | 214 (62%) | 160 (58%) | 258 (61%) | 113 (70%) |
| EXC | 118 | 78 (in body of text)<br>235 (at section ends)<br>150 (assignments) | 678 | 177 |
| MC | 293 (86%) | 249 (91%) | 394 (92%) | 161 (100%) |
| PC | 158 (46%) | 118 (43%) | 159 (37%) | 55 (34%) |
| CF | 30 (9%) | 8 (3%) | 10 (2%) | 0 (0%) |
| TL | 338 (99%) | 252 (92%) | 417 (98%) | 161 (100%) |
| EL | 37 (11%) | 39 (14%) | 48 (11%) | 18 (11%) |
| IM | 246 (72%) | 222 (81%) | 285 (67%) | 36 (22%) |
| IE | 136 (40%) | 92 (33%) | 134 (31%) | 18 (11%) |
| ER | 109 (32%) | 134 (49%) | 152 (36%) | 139 (86%) |

**N** is the total number of events analyzed for each document, excluding Exercises (EXC).

The values in Table 6 indicate that the three textbooks had similarities. For each book the percentage of events corresponding to the category codes CP, EX, MC, PC, CF, TL, EL, IM, IE, and ER were similar. The three textbooks had approximately a 60% to 40% breakdown between concept presentation (CP) and examples (EX). Over 85% of events used a mathematical context (MC), approximately 40% of events used a physical context (PC), and less than 10% of events were context free (CF). In addition, for each textbook, ***language use*** was at least 90% ***technical language*** and approximately 10% ***everyday language***. Mathematical knowledge as a *source of conviction* (IM) occurred in approximately 70% to 80% of events, while physical experience as a *source of conviction* (IE) occurred in approximately 30% to 40% of events.

The one difference that is apparent in Table 6 occurs in comparing the Alpha University lab manual to the textbooks. The lab manual generally used rules as a *source of conviction* (ER), rather than mathematics or experience (IM or IE). In the lab manual, 86% of events used rules as a *source of conviction*, whereas for the textbooks the corresponding figures are less than 50%.

## Summary

The philosophies of the three approaches to instruction as articulated by the instructors indicated similarities between the three approaches. Professor Alpha identified the development of problem solving skills as a key objective of the technique-oriented approach to instruction. Professor Beta described concepts-first instruction as a process of dealing with concepts in an informal way first to give students a basis from which to develop concept meanings and problem solving skills. Professor Gamma interpreted the infinitesimal approach to instruction as a means by which students learn how to think.

Technique-oriented instruction as delivered by Professor Alpha was organized and clearly and logically presented. Definitions, concepts, examples, theorems, and proofs were clearly identified and presented in a mathematically elegant and logical format. Concepts-first instruction as delivered by Professor Beta varied in format and the level of rigor incorporated into presentations. Throughout the term he incorporated informal and graphical interpretations of concepts, although there was an increased level of formal and rigorous presentations as the term progressed. Infinitesimal instruction as delivered by Professor Gamma was generally conducted in a questioning mode. Professor Gamma's instruction explicitly examined reasons for and connections between various aspects of the related mathematics. In addition, classroom observations indicated Beta College classroom instruction included more use of graphs and Gamma College classroom instruction included more use of *everyday language* and mathematics as a *source of conviction*.

The instructional materials for the three approaches to calculus instruction reflected few differences between the three approaches. The major distinction occurred with the Gamma College textbook. It contained more variety of exercise types, including exercises that require identification, creation, proof and explanation of mathematical events. These types of exercise were not present to the same extent in the exercise assignments for the other two approaches to instruction. Thus, it can be stated that the natures of the three approaches to instruction as reflected in the related instructional materials are similar, but the infinitesimal materials contain a different variety of exercises.

## C. Student Interviews

In this portion of the report the first three research objectives related to students' *language use*, *sources of conviction* and manner of construction of conceptualizations are addressed. Throughout the section examples of students' interview responses are given to illustrate a number of features and researcher interpretations of the responses. Seventeen students were interviewed. The seventeen students and the post-secondary institution each attended are given in Table 7. For reasons of confidentiality the names used for these students are pseudonyms.

**Table 7 - Interview Students and the Post-Secondary Institution Each Attended**

| Alpha University | Beta College | Gamma College |
|---|---|---|
| Annabel<br>Ellen<br>Jennifer<br>Ned<br>Richard | Cindy<br>Daniel<br>Doug<br>Leanne<br>Sally<br>Tim | Betty<br>Gordon<br>Mike<br>Nadine<br>Neil<br>Tanya |

### Completion Scores for the Clinical Interview Problems

Each student was assigned a "completion score" for the clinical interview problems. This score was determined by assigning a score of 0, 1, 2, or 3 to a student's problem responses according to the following criteria:

> 0 - The student was basically unable to begin to explain or solve the problem.
>
> 1 - The student gave a minimal response to the problem, either by giving a relatively short explanation, or proceeding through only the initial steps of a solution.
>
> 2 - The student gave a partial response to the problem and the response could be considered to be at least 50% complete. That is, the student's solution or explanation was at least halfway to completion. If it was not completed to this extent then a score of 1 was assigned.

3 - The student's response to the problem could be considered complete in that it included correct solutions or appropriate, self-contained explanations. For concept oriented problems a response did not have to include all possible explanations of a concept. Rather, the explanation had to include appropriate or varied explanations.

Students' scores on Problems 2, 3a, 3b, and 4 through 12 were totalled to obtain a student's completion score. Thus, each student received a score out of 36. These scores were used to determine an average completion score for each institution (see Table 8). Problem 1 (asking what calculus is all about) was not included in this scoring because it was more appropriately included with data from the personal interview questions. The average completion scores for the three institutions were similar, with the slightly lower score at Beta College likely due to the fact that students at Beta College were generally unsuccessful with Problem 10 (the skill problem requiring implicit differentiation). If the completion scores for all students are ranked, then students from each of the three institutions are found in the high-, mid- and low-ranges of the ranking. This contributes to the generalizability of the research results by demonstrating that students representative of a range of levels of calculus knowledge were interviewed at each institution.

It must be noted that a student's completion score should not be viewed as a measure of the student's achievement level in his or her calculus course. This precaution is because the calculus knowledge and skills examined by the clinical interview problems were limited in number and contained several problems unlike those generally found on calculus achievement tests. In addition, the clinical interview problems were not administered with the intent of being an achievement test, and students were aware of this fact. As well, since students were required to respond to the problems in an interview rather than purely written format, it would not be reasonable to interpret completion scores as standard achievement scores.

## Table 8 - Completion Scores for the Clinical Interview Problems

| Institution | Student | 2 | 3a | 3b | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total /36 | Rank | Institution Average |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alpha | Annabel | 3 | 3 | 2 | 3 | 2 | 2 | 3 | 1 | 3 | 3 | 2 | 2 | 29 | 5 | 25.4 |
| | Ellen | 2 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 17 | 14 | |
| | Jennifer | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 2 | 2 | 26 | 8 | |
| | Ned | 3 | 3 | 1 | 1 | 2 | 2 | 3 | 0 | 2 | 3 | 2 | 2 | 24 | 10 | |
| | Richard | 3 | 3 | 3 | 2 | 2 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 31 | 2 | |
| Beta | Cindy | 3 | 3 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 0 | 2 | 2 | 23 | 11 | 22.7 |
| | Daniel | 1 | 1 | 2 | 3 | 2 | 1 | 0 | 0 | 0 | 0 | 2 | 2 | 14 | 17 | |
| | Doug | 0 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 2 | 0 | 2 | 1 | 15 | 16 | |
| | Leanne | 3 | 3 | 2 | 2 | 2 | 2 | 1 | 3 | 3 | 1 | 2 | 1 | 25 | 9 | |
| | Sally | 3 | 3 | 2 | 2 | 3 | 2 | 3 | 3 | 3 | 0 | 2 | 3 | 29 | 5 | |
| | Tim | 3 | 3 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 2 | 2 | 30 | 4 | |
| Gamma | Betty | 3 | 2 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 0 | 2 | 1 | 17 | 14 | 25.3 |
| | Gordon | 3 | 3 | 1 | 2 | 2 | 2 | 1 | 1 | 2 | 0 | 2 | 2 | 21 | 12 | |
| | Mike | 3 | 3 | 2 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 31 | 2 | |
| | Nadine | 3 | 2 | 1 | * | 2 | 2 | 3 | 2 | 3 | 0 | 2 | 0 | 22 | 13 | |
| | Neil | 3 | 3 | 2 | 1 | 2 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 29 | 5 | |
| | Tanya | 3 | 3 | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 2 | 32 | 1 | |
| **Problem Averages** | | | | | | | | | | | | | | | | |
| | Alpha | 2.8 | 2.6 | 1.8 | 1.8 | 2 | 2 | 2.6 | 1.4 | 2.2 | 2.6 | 1.8 | 1.8 | | | 24.4 |
| | Beta | 2.2 | 2.5 | 1.8 | 1.8 | 2.2 | 1.7 | 1.8 | 2 | 2.2 | 0.7 | 2 | 1.8 | | | |
| | Gamma | 3 | 2.7 | 1.5 | 1.4 | 2.3 | 2.3 | 2.5 | 2.2 | 2.5 | 1.3 | 2 | 1.5 | | | |
| | | 2.6 | 2.6 | 1.7 | 1.7 | 2.2 | 2 | 2.3 | 1.9 | 2.3 | 1.5 | 1.9 | 1.7 | | | |
| | | | | | | | | | | | | | | | | |

* No score is entered for Nadine for Problem 4 because there was insufficient time in the interview to complete Problem 4. Therefore, her final completion score has been adjusted to be out of 36.

### *Sources of Conviction*

As mentioned in Chapter 3, the distinction between internal and external *sources of conviction* proved problematic during extensive examination of the student interview transcripts. Examples of these difficulties will now be described, along with reasons for the need to refine the related construct.

The original intention to classify students' statements as indicative of a particular *source of conviction* encountered difficulties when transcript excerpts such as the following were analyzed. The first excerpt is from Sally's response to Problem 3a, and the second is taken from Richard's response to Problem 9. Sally is from Beta College (concepts-first instruction) and she ranked fifth out of seventeen according to her Completion Score (see Table 8). Richard is from Alpha University (technique-oriented instruction) and he ranked second according to his Completion Score.

(Sally)
(Problem 3a)
[3. (a) Evaluate the following:

$$\lim_{x\to\infty} \frac{x^4 + 4}{x^3 - x + 5}]$$

$$\frac{(\infty)^4 + 4}{(\infty)^3 - \infty + 5}$$

**Figure 18. Sally's Written Response to Problem 3a**

I: You said you take the highest power. Can you say more about that?

S: Well in a polynomial like that. Like this is to the fourth degree. And this is to the third degree. But if this was to the fourth degree, then you could just take the fraction one to one. But it's not. It's to the third degree.

I: What if it was to the third on the top and the fourth on the bottom?

S: Then this one would always be bigger. So then the limit would be zero.

I: What else was it you were going to say?

S: Just because when you divide a small number by a larger number it will always get slowly closer to zero as the larger number gets larger.

I: Now you said back at some point that this was a rule you were taught. To you is it a rule or do you have a way of justifying it?

S: Well to me it's a rule, but I guess it always works. Like he proved it does work. So it's not just a saying, but something you can use to solve a problem.

(Richard)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]

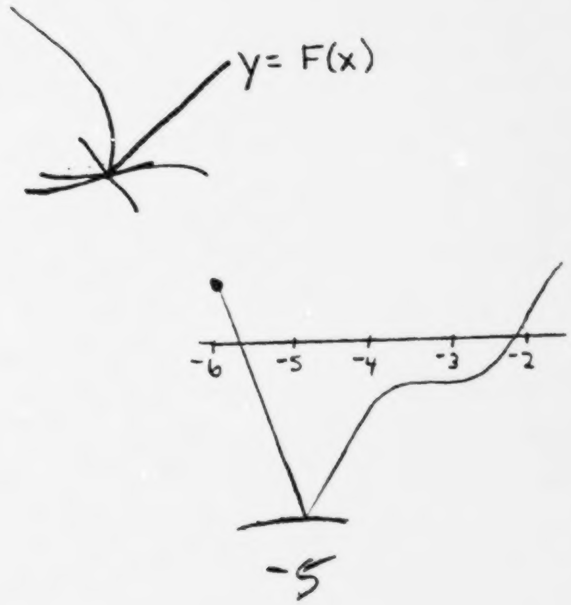


Figure 19. Richard's Written Response to Problem 9

R: Well, there's no way to measure the slope of the tangent for these lines [at x = -5]. Like how can you do the slope of this line. How can you? And then on the five. There's no way. There's no line there. It breaks. In these two lines it breaks, so there's no way you can measure the slope of the function at that point. And then this point, because it comes to a cusp. There's no way to measure the derivative. It's not a smooth curve. You can't measure the derivative of that.

I: And why not?

R: Well I mean this is the way a cusp is drawn. But if you could get a cusp it could go like that [draws a cusp]. And you don't know what, can you draw like that? How do you draw, you can draw it a lot of different ways.

I: If I asked you to prove to me algebraically there isn't a derivative there, what would you put? You are able to explain, but could you write it down symbolically?

R: I wouldn't even know where to start.

I: Could you do it here where it jumps?

R: Well you know it's not continuous. It's not continuous, so there's no derivative.

I: But do you know why that is?

R: No. I know the rules. I just don't know why.

Both these interview extracts display a variety of *sources of conviction*. Sally and Richard both refer to "following rules", yet they also give valid explanations for their decisions. They appear to hold rules as a *source of conviction*, yet they simultaneously display use of mathematical or experiential (visual) knowledge as a *source of conviction*. The difficulty in categorizing *sources of conviction* arose not from a display of more than one *source of conviction* in a particular problem response, but in what that multiplicity indicated for other responses. For example, throughout their problem responses Sally and Richard displayed mathematical and experiential *sources of conviction*. In analysis of specific problem responses this fact initially led the researcher to conclude Sally and Richard each used a high degree of internal *sources of conviction*. However, this conclusion had to be questioned when one considers that, alongside use of correct procedures and explanations, Sally and Richard often mentioned "following rules". This mention of "following rules" implies that although their responses demonstrated valid, correct use of mathematics, Sally and Richard did not perceive their responses as originating from themselves. Rather, they saw them as arising from knowledge of externally generated rules.

As in the two interview extracts already given, a display of mathematical or experiential knowledge often coincided in Sally and Richard's interviews with "following rules". This fact seemed to indicate that *sources of conviction* arising from mathematical or experiential knowledge were not necessarily perceived by Sally and Richard as internal, personal knowledge. Thus, the assumption that mathematical or experiential *sources of conviction* are necessarily internal had to be revised. The need for this revision was further demonstrated in the personal interviews with Sally and Richard, where they gave responses such as:

> (Sally)
> In psychology you see things for yourself. But in science you just follow rules. And there's not, well there is proof, definitely. But you know, there's all rules about how things work and why things work. And I tend to see things in a more scientific way. And then in math I work with rules. And how things work and what he says I just take as how you do it, you know.

> (Richard)
> Well I mean just that in calculus when I say I understand something it means I can do it. And I can get the right answer. Whereas if I'm usually talking about another subject, well I understand the theory or I understand the principles behind it. I know what is happening and I could, if somebody asked me to explain it to them I could explain it to them in terms they could understand. Whereas I couldn't do that at all, I could never explain calculus to somebody in terms that they could understand. Because I don't understand. I just know how to do it.

Sally and Richard's words reveal that although they were able to explain the mathematics of many of the clinical problems, they had not actually "internalized" the corresponding mathematical or experiential *sources of conviction.* That is, they did not speak of mathematical knowledge as residing within themselves. They appeared to have fairly coherent calculus conceptualizations, but did not necessarily view these structures as internal, personal constructions. Instead, these structures were viewed as reproductions of externally generated knowledge.

Although Sally and Richard's interview responses have been used to demonstrate a need to refine the initial formulation of the concept of *sources of conviction*, the nature of their responses was not unique. Similar apparent inconsistencies occurred in most of the interview transcripts. Comparison of a student's explanations of calculus ideas and what he or she said about his or her calculus learning revealed the original *sources of conviction* categories could not be consistently applied to student interview data. In particular, mathematical and experiential knowledge could not necessarily be interpreted as internal *sources of conviction.*

It therefore became apparent that the *sources of conviction* a student displayed on the surface of a specific problem response did not necessarily reflect the overall picture of her or his *sources of conviction.* As a result, the nature and role of a student's *sources of conviction* had to be determined from the overall picture of his or her *sources of conviction* as displayed in both the clinical and personal interviews. Results of the overall analyses revealed three main groups of students, with students in one group similar in the ways they perceived and justified mathematics. Before these groups are described fully, the emergence of distinctions between various students' *sources of conviction* will be discussed.

*Sources of conviction*, as conceived of in this study, refer to where an individual sees truth and validity residing within the context of learning or using calculus. More specifically, *sources of conviction* refer to how students determine what are legitimate or correct mathematical ideas and procedures. According to constructivism, if learning is an adaptive process through which an individual constructs a viable model of the world, then the *sources of conviction* by which this construction occurs are influential components of what is learned. That is, the nature of what a student learns will be influenced by both the nature and role of the individual's *sources of conviction*. For example, a student whose primary *source of conviction* is a collection of disjointed, memorized rules or procedures is likely to perceive his or her mathematics learning differently than a student who sees a coherence or structure to calculus concepts, rules and procedures. The two students might justify their problem responses by reference to similar mathematical rules or interpretations, but the nature and role of their *sources of conviction* could be quite different. The nature of their calculus conceptualizations would therefore also be different. This situation is exemplified in the following three sets of extracts from the interviews with Doug, Jennifer and Tanya. Doug is from Beta College, Jennifer is from Alpha University, and Tanya is from Gamma College. In terms of Completion Scores, these three students ranked sixteenth, eighth and first, respectively (see Table 8).

(Doug)
(Problem 3a)
[3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}]$$

I remember he said something about this. If there are different powers. If the top one is a different power. If I was just to say what I think it would be, I think it would be infinity. But I don't know.

(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

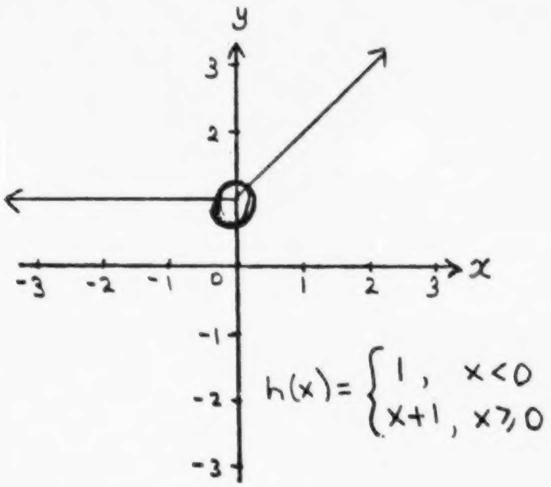


**Figure 20. Doug's Written Response to Problem 5**

I just remember in class, you know, that if it breaks like that it's not continuous. . . . I guess cause it changes direction. It goes this way and then it breaks that way.

(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?]

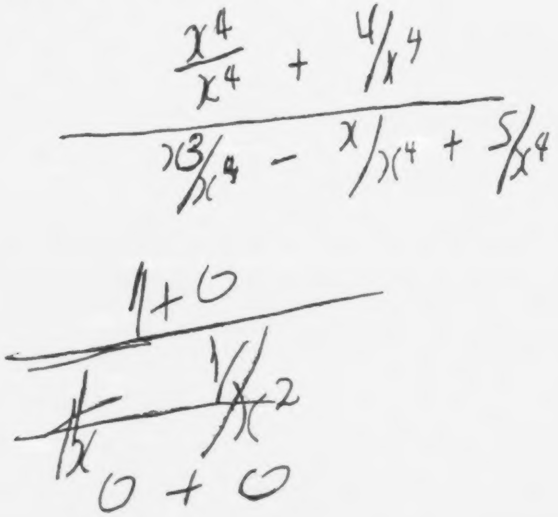
Well I suppose I'd rather know why. I'm not sure I do though. Like I could say I remember in class that if you have this situation there's no derivative, but I don't know why there's no derivative. . . . It's okay if I get it right on the test.

(Personal Interview)
I just do it. I just do it because I don't understand it. I just do it and I do good. I got 76 on the last test but I don't understand anything I did. I just memorized how to do things.

Cause it's more, all I'm doing is memorizing his examples. I'm not really understanding what's going on.

In the above extracts Doug's response to Problem 3a is mathematically incorrect in that association of the word "continuous" with the word "break" has led him to construct a mathematically incorrect conceptualization of continuity. These extracts also show that Doug's *sources of conviction* are external in nature, governed by what he remembers of what was said in class by the teacher. The next extracts, from the interview with Jennifer, will show Jennifer's *sources of conviction* are similar to Doug's in that they are based on knowledge of rules and procedures. However, the extracts will also show her *sources of conviction* are different than Doug's in that Jennifer perceives rules and procedures to be processes that "make sense".

> (Jennifer)
> (Problem 3a)
> [3. (a) Evaluate the following:
>
> $$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}]$$



$$\frac{\frac{x^4}{x^4} + \frac{4}{x^4}}{\frac{x^3}{x^4} - \frac{x}{x^4} + \frac{5}{x^4}}$$

$$\frac{1+0}{\frac{1}{x} - \frac{1}{x^2}}$$

$$0 + 0$$

**Figure 21. Jennifer's Written Response to Problem 3a**

> I: And why is it you did this very first step? You took this and you divided it by x to the fourth. Why did you do that?
> J: Just ah. [pause] I don't know. It's just something I've been taught. . . . It's just, well for certain things. There's just, like there's certain rules which are just rules. And you can just build from there. But I wouldn't know how to get those basic rules.
>
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
>
> Continuous would be meaning that the graph, well we were always taught that if it's continuous you don't have to lift the pencil from the paper. That there would be no breaks in the graph.

(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]

Well at that point it's got, it's just a single solitary point [at x=-6]. It's not really a function. It's not like an entire expression. So it's like y equal to one or whatever. It's not a point. Or, it's just a point. It wouldn't have like a slope or anything like that.

(Personal Interview)
I think calculus, if you get into a method of thinking, it's just a process. It seems to be the same sort of process, and you just get into that method of thinking and it's all very logical.

There's certain ones, you know, these are rules and okay that's great, I'll just follow these rules.

This will sound kind of weird, but I find calculus, it's just sort of a way of thinking. Then if you can establish that sort of process, then things just seem to make sense.

In the above extracts Jennifer's perceptions of her calculus knowledge reflect *sources of conviction* that are more internal in nature than Doug's. They are more internal in that Jennifer perceives her calculus knowledge as a "method of thinking" that is "logical" and therefore personally understandable. The next extracts will show how a third student, Tanya, uses *sources of conviction* that have both similarities and differences to Doug and Jennifer's *sources of conviction*.

(Tanya)
(Problem 3a)
[3. (a) Round off the following:

$$\frac{M^4 + 4}{M^3 - M + 5}]$$

$$\frac{\frac{M^4}{M^4}+\frac{4}{M^4}}{\frac{M^3}{M^4}-\frac{M}{M^4}+\frac{5}{M^4}} = \frac{1+\frac{4}{M^4}}{\frac{1}{M}-\frac{1}{M^3}+\frac{5}{M^4}} = \frac{1+dx}{dx}$$

$$\rightsquigarrow \infty$$

**Figure 22. Tanya's Written Response to Problem 3a**

Because this would be an indeterminate amount. You can't really see what's happening because this is infinity over infinity. And that really doesn't say anything. . . . So I took a factor of M to the fourth. I could have taken a factor of M to the third, but I probably would have had to simplify one more time. I did M to the fourth because it's the largest factor in there, to just simplify it. I simplified it and then just rounded off. Once again dx stands for an infinitesimal. Any finite over an infinite is an infinitesimal.

(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

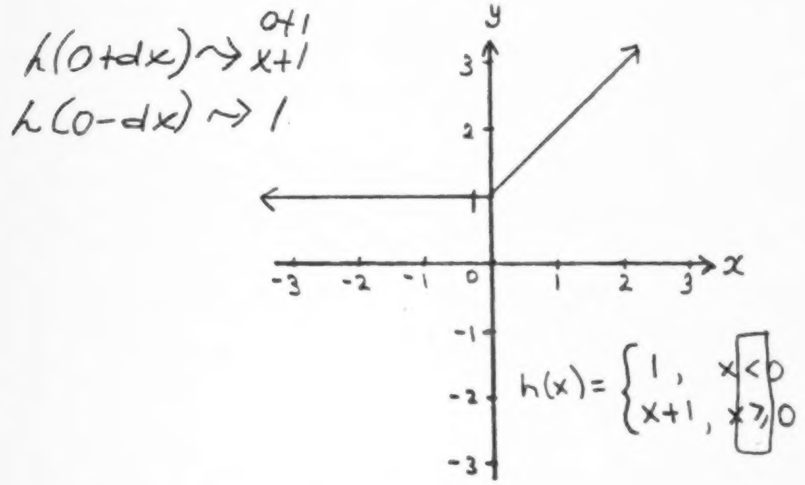


**Figure 23. Tanya's Written Response to Problem 5**

T: Well it joins up everywhere. It, the normal English definition of continuous is you don't have to lift your pen off the paper. And you can see here you don't. And once again this definition applies to this too.
I: How would you deal with this definition right here where it does join? [at x=0] In other words, if I asked you to prove it's continuous at zero, could you do that algebraically?
T: Well, um. I would have to look at the point left, and a point to the right of zero. Right at zero there's a point and it's one. Right at x equal zero there's a point. . . . This is a function by itself. Even though it is a split function it's still a function. Because it connects right at zero. You can see by the signs. This is less than and equal to. This is a less than. Here it connects with the same idea I was saying right here. Take an infinitesimal point right to the left. It will round off to the function itself at x.

(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?]

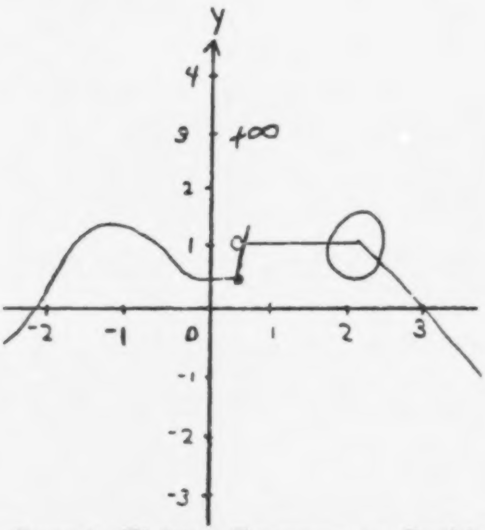


**Figure 24. Tanya's Written Response to Problem 9**

Ah. The derivative. The derivative again is an infinitesimal change in x. If I take the point right here [at $x = \frac{1}{2}$]. And a point just to the right of it. Going from here right up to here. And that is positive infinity. It's a straight line. The derivative of a straight vertical line.

(Personal Interview)
So to me to understand calculus is very important. I enjoy it. And I think it's fascinating. You need an imagination for it. . . . This rate of change of certain things. And all this whole business, You need an imagination. You need an imagination, not only on paper, but you have to kind of see what happens. What's happening at a certain time. You need to see that. And yet a lot of what is going on in calculus, with infinity and infinitesimals, and adding them and subtracting them. Sure you can do that on paper, but you kind of have to see what goes on. You kind of have to imagine that these sequences keep on going. They just don't stop. So you need an imagination for it.

Because you can't learn by memorizing everything. Because you have to interpret it. And you have to understand the theory behind a certain form. The theory behind a certain something, and then apply it to something else.

Well from unit to unit you fit together everything that you learned before. Like limits apply to hyperreals, and derivatives apply to hyperreals. Everything you learn applies to infinitesimals and infinites. It all fits together.

These interview excerpts show Tanya's *sources of conviction* are similar to Doug and Jennifer's in their inclusion of knowledge of rules and procedures. However, they differ as to how Tanya perceives and uses them. Tanya perceives of calculus as a body of knowledge that "all fits together". She sees herself as an important factor in this fitting together in that she can use of her own imagination to "see what happens" when she works with calculus concepts, rules and procedures. Tanya's *sources of conviction* can therefore be said to be highly internal in nature in comparison to either Doug's or Jennifer's. They are more internal in that Tanya acknowledges and makes use of what she perceives to be her own, personal understandings of calculus. She expresses a belief that for her to learn and apply calculus concepts she must "interpret" and "understand the theory behind a certain form". That is, Tanya sees herself as an important component of learning calculus. In comparison, Doug and Jennifer do not explicitly acknowledge their own understandings as essential factors of calculus learning. Doug speaks of his calculus learning as separate from his understandings, and for him calculus learning is being able to "just do it". Finally, although Jennifer speaks of her learning as a "method of thinking", she does not state she sees herself as a contributing component of this "method of thinking". Rather, she sees the knowledge of "certain rules" as a "very logical" process one employs to do calculus tasks.

Thus, in terms of the role of a student's *sources of conviction* in the construction of calculus conceptualizations, it can be said that the internal nature of Tanya's *sources of conviction* guide her to construct knowledge over which she feels she has personal understanding and control. At the other extreme, Doug's *sources of conviction* are highly external in nature and they play a role in his calculus learning by leading him to construct knowledge he feels he can use but cannot understand. There is a sense that Doug feels he can control his calculus knowledge to "do good" on tests, but there is no sense Doug feels his calculus knowledge is due to his own personal constructions. Rather, Doug's calculus knowledge appears as a reproduction of what he remembers the teacher said. Jennifer's *sources of conviction* are in between Doug's and Tanya's in the extent to which they are external or internal in nature. Her *sources of conviction* are external in their inclusion of knowledge of externally generated "basic rules". They are concurrently internal in that Jennifer can use "basic rules" to "build" her problem responses. The role of her *sources of conviction* can therefore be said to be as tools from which she can "establish" a "sort of process" or "method of thinking" as a technology by which one does calculus tasks. Then, Jennifer can be viewed as a technician of the application of calculus ideas and rules. Lastly, Tanya's appreciation of calculus and desire to understand it reflect *sources of* *conviction* that are internal in nature and internal in the role they play in guiding Tanya to feel personal understanding of the calculus conceptualizations she constructs.

Doug, Jennifer and Tanya are representative of the three main groups of students that emerged from examination of the clinical and personal interview responses. Each of these three groups are discussed in detail in the upcoming sections of this chapter. The three groups are referred to as Collectors, Technicians, and Connectors. The groups differ from each other in the degree to which their *sources of conviction* are external or internal in nature. The nature of their *sources of conviction* creates differences between the three groups of students in the role their *sources of conviction* play in construction of calculus conceptualizations. Collectors exhibit the highest degree of externalized *sources of conviction*, while Connectors exhibit the highest degree of internalized *sources of conviction*. Technicians fall somewhere in between these two other groups, exhibiting a mixture of external and internal *sources of conviction* (see Figure 25).

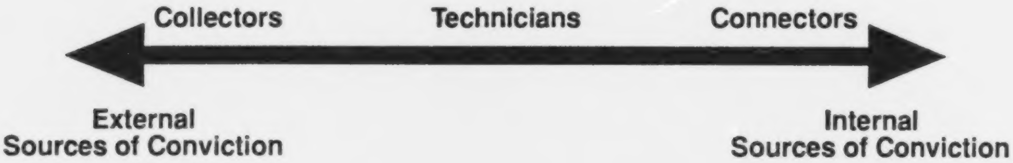


**Figure 25. The Nature of Collector, Technician and Connector *Sources of Conviction***

The names for the three groups of students reflect the nature and role of the groups' *sources of conviction*. Collectors assemble a set of memorized statements, rules and procedures. Technicians organize a set of statements, rules and procedures that can then be logically employed as a technique for thinking about and applying calculus concepts. Connectors also organize statements, rules and procedures, but their related conceptualizations are inter-connected, personally understandable structures and processes. The characteristics of the three groups are such that they can be viewed as embedded one inside the other. More specifically, Technicians exhibit *sources of conviction* similar to Collectors' *sources of conviction*, but they also exhibit *sources of conviction* distinctly different from those of Collectors. Similarly, Connectors exhibit *sources of conviction* similar to those of Collectors and Technicians, but they also display *sources of conviction* that differ from those of Collectors and Technicians. Connectors' *sources of conviction* can therefore be said to envelop Technicians' *sources of conviction*, which in turn envelop

Collectors' *sources of conviction*. Table 9 indicates the classification of each of the 17 interview students in terms of his or her *sources of conviction*.

**Table 9 - Classification of Collectors, Technicians and Connectors**

| Collectors | Technicians | Connectors |
|---|---|---|
| Ellen (Alpha)<br>Ned (Alpha)<br>Cindy (Beta)<br>Daniel (Beta)<br>Doug (Beta)<br>Leanne (Beta)<br>Betty (Gamma)<br>Gordon (Gamma) | Jennifer (Alpha)<br>Richard (Alpha)<br>Sally (Beta)<br>Nadine (Gamma) | Annabel (Alpha)<br>Tim (Beta)<br>Mike (Gamma)<br>Neil (Gamma)<br>Tanya (Gamma) |

### Collectors

The students classified as Collectors are Ellen, Ned, Cindy, Daniel, Doug, Leanne, Betty and Gordon (see Table 9). Ellen and Ned are from Alpha University, Cindy, Daniel, Doug and Daniel are from Beta College, and Betty and Gordon are from Gamma College. These 8 students ranked fourteenth, tenth, eleventh, seventeenth, sixteenth, ninth, fourteenth and twelfth according to Completion Scores (Table 8). A student who from his or her *sources of conviction* is classified as a Collector displays *sources of conviction* that are generally external in nature. These *sources of conviction* are external in that they reside in statements, rules and procedures presented by the teacher or textbook. They do not generally reside in what the student has construed for herself or himself. The student constructs his or her mathematical knowledge by assembling isolated, relatively unconnected mathematical statements, rules and procedures. Thus, the student's calculus conceptualizations can be said to be a "collection" of statements, rules and procedures. More specifically, the external nature of a Collector student's *sources of conviction* guides the student to approach calculus learning as recall or rote memorization of statements, rules and procedures. In this way the role of a Collector student's *sources of conviction* is as a validation to the student that he or she makes statements and performs procedures that will be recognized as valid or correct by other individuals. Although the student might validly apply calculus knowledge, the student does not claim to know personally whether particular pieces of mathematics are valid or correct. Rather, the student relies on others to determine validity or correctness. These other individuals are perceived by the student to be people for whom calculus is understandable and meaningful.

A distinctive feature of Collector students' *sources of conviction* was their external nature. Collector students frequently referred to statements, rules and procedures that had been given by the teacher or textbook. They said such things as:

> (Doug)
> Like I could say I remember in class that if you have this situation there's no derivative, but I don't know why there's no derivative.
>
> (Cindy)
> And the way to do that one was he drew a line like this, a secant line. . . . And so he's making this point here closer and closer to this. And then he takes. [pause] Oh, I can't remember now.
>
> (Ellen)
> I don't know if this is right, but I think I remember something like that from the textbook.
>
> (Gordon)
> I don't know why. I just remembered something about there's not a derivative.
>
> (Betty)
> I think one time he said in class just to, like um, he said something about letting infinitesimals and infinites because, um. What was it? [pause] Like I don't understand why we have to round off.
>
> (Ned)
> Well I was just taught that in this course this year. What I would have done before is I would have used trial and error. Before this class.
>
> (Leanne)
> I: Can you show me on the diagram what it involves?
>
> L: Um. Probably not. Like if I have an example beside me to follow, then I can do it.
>
> (Daniel)
> I remember reading in my calculus textbook something about continuous functions are smooth with no breaks. And I was trying to remember if it also included, what do you call it, ah, sharp turns in a graph. Because I believe that might make it discontinuous at this point right here.

In all the above interview extracts the students make reference to what they remember from their class or textbook, using the teacher or the textbook as a *source of conviction*. These *sources of conviction* are external in nature in that the students employ them to reproduce what they remember from class or the textbook. More explicitly, the students use the teacher or textbook as a means of validation, while they concurrently state they either "don't know" or "can't remember" why a particular piece of mathematics is as it is. That is, the students do not claim any ownership of the calculus concepts, rules or procedures they use. By the external nature of their *sources of conviction* they have built calculus conceptualizations that are an assemblage or collection of externally generated statements, rules and procedures.

The fact that the students each give some sort of response to the calculus problems indicates they have constructed some sort of calculus conceptualizations, even if they do not feel a personal sense of comprehension of these conceptualizations. At times however, as with Daniel's misconception that "sharp turns in a graph" constitute discontinuities, it is clear that Collector students construct conceptualizations that are viable in terms of their understandings of calculus language. That is, language knowledge serves as a *source of conviction* upon which to build calculus conceptualizations. For Collector students this language knowledge often leads to construction of mathematically incorrect conceptualizations, as it did with Daniel's association of the words "smoothly" and "no breaks". Doug displayed a similar misconception (see page 138). Thus, while constructing calculus conceptualizations as a collection of externally generated statements, rules and procedures Collector students are also engaged in personal calculus interpretations. These interpretations are constructions that are generally internal in nature, arising from language knowledge as a *source of conviction*. Additional examples of Collector students' *language use* as an internal *source of conviction* upon which to build calculus conceptualizations are the following:

> (Ned)
> (Problem 3b)
> [(b) What does "limit" mean to you?]
>
> But um, so you have to really assume all limits. Say if you're drinking you should know your limit. Well, it's so uncertain, but you should get an idea of what your drinking limit is. It's very independent.

(Daniel)
(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

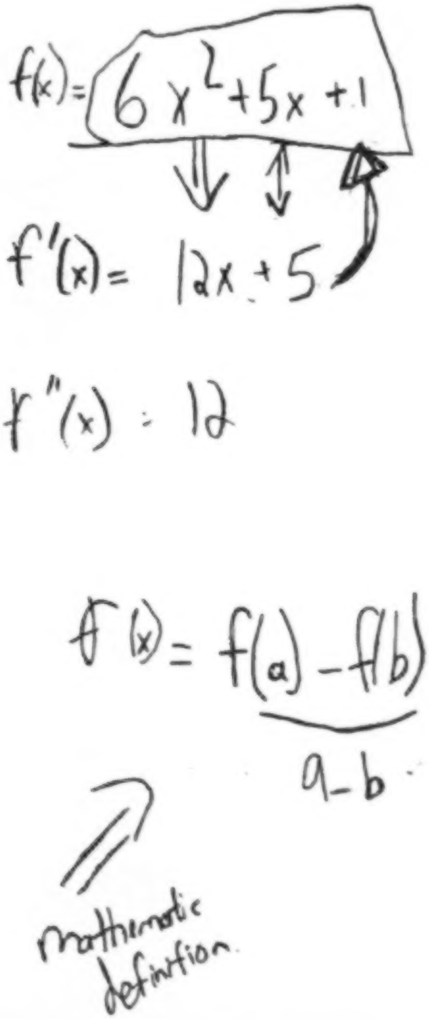

f(x) = 6x² + 5x + 1
f'(x) = 12x + 5
f''(x) = 12
f'(x) = f(a) − f(b) / a − b
mathematic definition

**Figure 26. Daniel's Written Response to Problem 6**

I'd say that a derivative may be along the lines of a root of something like this. . . . Like a derivative to me is like smaller, it's like somehow made into this. Like it's like a factor of this, but it is not this whole expression. It's more like a root, a root of this expression. . . . But it is to me like ah, I guess root is a word I'd ascribe to this, other than derivative.

(Ellen)
(Problem 2; second sequence)
[2. For each of the following sequences of numbers, decide whether the sequence has a limit. If so, what is this number?

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$$

3.9, 3.99, 3.999, 3.9999, 3.99999, 3.999999, . . .]

No. No limit. . . . It just seems to get bigger.

(Cindy)
(Problem 5)
(for the graph for Problem 5 refer to page 61)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

Well, it does because that's what continuous means. It means continuing on and not having a break. That's what continuous means. That you don't make a break in something. And so here you're not making a break [first graph]. And here you're not [second graph]. And here you're making a break, jumping [fourth graph]. And actually this one here has two different definitions for the graph [third graph]. And so obviously your graph isn't continuous if you have two different definitions.

In the above excerpts Ned and Ellen construct conceptualizations of limits in terms of their previous knowledge, respectively, of the words "limit" and "bigger". Their language knowledge has served as a *source of conviction* upon which they have constructed their problem responses. Similarly, Daniel uses the word "root" to describe the derivative. He thereby has associated with the derivative a number of his notions of the word "root", including the notion that roots are "smaller" than the whole and are "somehow made into something else". Cindy uses her understandings of "two pieces" to construct a misconception that the third function of Problem 5, $y = H(x)$, is discontinuous. In all cases the students use previous *everyday language* knowledge as a *source of conviction* upon which to build calculus conceptualizations. This use of *everyday language* as a *source of conviction* reflects Halliday (1978), Pimm (1987) and Ernest's (1991) notions of the importance of language in mathematics learning. They argued that an individual's use and interpretations of everyday language are likely to figure prominently in the individual's mathematics learning. This prominence is because it is through language use that "individuals construct subjective theories or personal representations" (Ernest, 1991; p.72). Johnson (1987) also argued that an individual's previous language meanings are important to construction of conceptualizations. He discussed how through development of image schemata individuals use bodily experiences and related language understandings as metaphors by which to develop language meanings. The role of language as a *source of conviction* is a key finding of this research study, and is discussed more thoroughly in a later section of this chapter.

Another feature of Collector students' interviews is that Collector students were often unsuccessful in completing the clinical interview problems. Forty of the 108 problem responses made by Collectors received a 0 or 1 Completion Score, which corresponds to no response or a minimal response (Table 8). In comparison, the other eight students received Completion Scores of 0 or 1 on only 7 of 95 problem responses. Collectors frequently made errors, displayed misconceptions, were unable to remember particular rules or procedures, or were unable to explain concepts. This lack of success is evidenced by the Collector students' Completion Scores for the clinical interview problems (see Table 8). Students classified as Collectors received Completion Scores in the range of 14 to 25. All other interview students, except Nadine, received completion scores higher than 25. Examples of Collector students' errors, misconceptions, failure to remember procedures, or failure to explain concepts are the following:

(Cindy)
(Problem 4)

[4. What can you say about the function $y = \dfrac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?]

Well, it's undefined. Because there's a zero on the bottom.

(Doug)
(Problem 4)
It's undefined. . . . Because you can't divide something by zero.

(Problem 7b)
[$F(t) = (2t^2 + 3t - 2)^{10}(3t^{1/4} - 9)^7$ ]

$$10(2t^2+3t-2)^9\left[(3t^{\frac{1}{4}}-9)^7\right]+(2t^2+3t-2)^{10}\left[7(3t^{\frac{1}{4}}-9)^6\right]$$

**Figure 27. Doug's Written Response to Problem 7b**

(Ellen)
(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

E: Well I could show her how to find one, but I can't, I don't, like I know how to find a first or second derivative, but I don't know why you do it.
I: Do you have any way of picturing them, or graphing them? What derivatives are about.
E: [long pause] No.

(Gordon)
(Problem 6)

$$y' = \frac{F(x+\delta x - F(x)}{\delta x}$$

**Figure 28. Gordon's Written Response to Problem 6**

I: Can you show me where it comes from? Is there any way it relates to this picture you've drawn with the tangent line?
G: I haven't got a clue. [pause] Basically it's just saying f at x plus dx minus f at x. It just gives you a dx. That's the way I look at it anyway. If you had something which was f at x and you add dx to it, that's just so small that all you have left is the dx.

(Betty)
(Problem 6)
I don't even know what a derivative is all about. . . . Something about a graph. Um. This is the unit I was just robotic in. This one, like the slope. To figure out what the slope of a graph was, you just use the double derivative.

(Ned)
(Problem 3b)
[(b) What does "limit" mean to you?]

It's yet to be determined. Like ah, if you find a pattern for $\pi$ for example. It might come out as a definite number. So I don't think a limit can be, like by definition you can't find a limit.

(Problem 4)

[4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?]

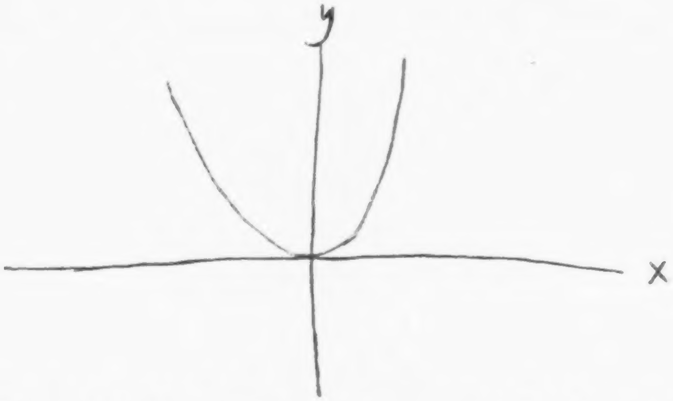


Figure 29. Ned's Written Response to Problem 4

I would take something of a parabola form as x squared. I draw it in.

(Leanne)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

I: Suppose I gave you just this. y equals x squared. And said, is that continuous or not? What would you say?
L: Yes. For whatever y value you give me I'll give you an x squared value.

(Problem 7a)

$$[y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}]$$

$$\frac{3x^2 + 1x^{-1}}{x^{1/2} + 6x \cancel{+7}}$$

$$\frac{3x^2 + (-x^{-2})}{x^{1/2} + 6x} = \frac{3x^2 \cancel{+} - x^{-2}}{x^{1/2} + 6x}$$

Figure 30. Leanne's Written Response to Problem 7a

These interview extracts demonstrate that Collector students frequently did not completely or correctly remember such things as the product rule, the quotient rule, or the chain rule. The students also displayed a lack of ability to explain derivative or limit concepts. Finally, as has already been discussed, Collector students displayed a number of misconceptions. These included the notion that any rational expression whose denominator is zero is undefined (Cindy and Doug), a non-rational number might have a repeating decimal expansion (Ned), the graph of a function that contains an $x^2$ is a parabola (Ned), and a function is continuous if it is defined everywhere (Leanne).

Another prominent aspect of Collector students was a belief that mathematics is a collection of definite, correct formulas, rules and procedures. Some examples of what the students said in relation to this aspect are:

> (Daniel)
> (Problem 3a)
> [3.(a) Evaluate the following:
>
> $$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}]$$
>
> Plus four. Which just equals a large infinite number. And here, infinity to the third minus an infinite number is still a large infinite number. The other way this problem can be done. Well not the other way, the correct way. . . . I know there's another way that's the correct way to do the problem. I know there's a correct way to do it, but I'm not remembering.
>
> (Doug)
> Well math kind of is black and white. . . . I just get to the end and I don't know if it's right or wrong. And in English, you know, if I read something and he asks a question on it, you just know it. You know it's got to be right.
>
> (Ned)
> We've already had hundreds of formulas tossed on us. An it's just an introductory course. There's got to be an infinite amount of formulas.
>
> (Ellen)
> Well I think that you're told to do this and this, you know. You just have to do it I guess. I don't know.
>
> (Gordon)
> Yeah, I got the steps. Here, here and here. What I do is write down the steps for every one.
>
> (Leanne)
> Just step by step on how to do a problem is what I understand best.

> (Cindy)
> That makes me think I'm wrong. But I think because there was some rule he gave us about taking the largest power over the largest power and using that.

> (Betty)
> But there's a definite answer to everything in math. That's why I took math.

Included in the above excerpts are views that mathematics consists of "steps" or rules to do "this and this", mathematics is "black and white", and mathematics has definite answers and "correct" ways that problems are to be solved. These views further reveal the external nature of Collector students' *sources of conviction* in that they show how Collector students perceive truth and validity decisions in mathematics as pre-determined, external entities. Collector students do not generally see it possible that these decisions might be influenced by one's own perceptions or interpretations. Rather, they must be remembered or memorized. In fact, Collector students explicitly state they approach their calculus learning by memorization of what they believe will be needed to pass an exam. The following interview extracts provide evidence of how Collector students view their calculus learning as memorization:

> (Gordon)
> Yeah. I didn't fully understand it. I was just memorizing. Like memorizing what to do with the formulas, but not understanding why you have to do it. Like if you get the slope, like the chain law, I know what to do to find the derivative. But I don't understand how that works out.
>
> But if I don't understand what he's done and why he's done it then I just memorize.
>
> I'm relying on being told that this is how everything works. . . . But this is just something I need so I just do it. Accept it and leave it.

> (Ellen)
> It's probably because I just memorized it to get me through that part.
>
> Like I know how to find a first or second derivative, but I don't know why you do it.
>
> I don't know why. I just remember learning it.

> (Ned)
> Just memorizing the formulas and stuff really helps me out a bit.
>
> Like you know, I can memorize all sorts of stuff, doing algebra and simple trig functions and stuff like that.
>
> (Doug)
> Well I just plain don't understand calculus. Like I get the questions, but it's not because I understand them. It's because I just memorized them. . . . But in other subjects, like anything, you read and you understand it. Like if you get an economics concept, you just understand it.
>
> I just do it. I just do it because I don't understand it . I just do it and I do good. I got 76 on the last test, but I don't understand anything I did. I just memorized how to do things.
>
> Like I could say I remember in class that if you have this situation there's no derivative. . . . That's okay if I get it right on the test.
>
> Because I don't really understand that. The basis of why I'm doing it. I just do it.
>
> (Daniel)
> Like in math because I find it difficult I try to memorize different ways of doing it.
>
> I have just memorized it and I use it. I don't particularly understand it.
>
> That's what I'm going to do for this test coming up on Thursday. I'm going to go through my theorems and everything that's been given us. And I'm going to go through the practice exam and I'm going to do all the types of questions he could ask.
>
> And right now it's just a matter of being able to produce it on a test.

In the above excerpts Gordon, Ellen, Ned, Doug and Daniel explicitly state they use memorization as a learning technique in calculus. They speak of "just memorizing" formulas and examples, and they make it clear they do not feel they personally understand calculus in terms of why one uses particular procedures, or how procedures function in reaching a solution. For example, Gordon speaks of not understanding "why you have to" use particular formulas, Ellen speaks of not knowing why one finds a first or second derivative, Ned comments that he is "just memorizing the formulas", Doug says he doesn't understand "the basis of why" he proceeds in certain ways while solving calculus problems, and Daniel says he doesn't "particularly understand" calculus. The other three Collector students, Cindy, Leanne and Betty, did not as explicitly use the word "memorize" in speaking about their calculus learning, but they spoke of their calculus learning in ways similar to the other Collector students. They said such things as the following:

> (Cindy)
> I can kind of work out formulas and work out a way, kind of memorize almost a way that he tells us to do it. But I don't really understand it. . . . I can understand, like I can memorize like derivatives. But I can't actually, there's a lot actually I don't understand. That I know for sure in my mind. Like I can memorize the problems he gives us. The types of problems. But if he goes and gives us a different type of problem on the exam, then I have a tough time with it.
>
> Oh I would like to be able to understand. But it would take time. A lot more time than I have to put into it.
>
> I tend to want to have everything in black and white where I can see exactly, like almost have a picture, a visual picture of what's going on. And sometimes I have to accept that I can't totally grasp it. There's a lot of things that we're doing now that I feel I can't totally grasp like that.

> (Leanne)
> Kind of from repetition I know what to do with it.
>
> But it's remembering which step and remembering the process to go through to get a thing. . . . Perhaps in calculus class I'm following that right now. But usually I like knowing. So I'm knowing I'm getting the right answer.
>
> I can do it but I don't know why. Just do it because he says so.

> (Betty)
> Well when I learn in class it's just, it's like a carbon copy. It's like, okay I know that. It's just because it's said so.
>
> And then you have to look back, and then you know it's all carbon copy from there. You just have to say, okay they did it like this, so I'll do it like this. But you don't know anything.
>
> When you understand something, like understand the whole, the basis of what we're doing in class. Like why do you need a derivative? Where does derivative come from? What, like what's the most important thing about a derivative? And basically I don't know what the most important thing is about a derivative. I know where it comes from, but it has no relation, like it has no meaning to me. And I have to have the meaning. . . . Because it will stay with you all your life if you understand it. For robotics it's just like memorization type thing, and I'm not very good at that.

In the above excerpts Cindy speaks of "kind of" memorizing ways to do calculus problems, Leanne talks about "repetition" or "remembering" of steps, and Betty speaks of a "carbon copy" or "robotic" approach to her calculus learning. However, although these students speak of their calculus learning in terms similar to the other five Collector students, they also speak of a desire to have a better understanding of calculus. Cindy says she "would like to be able to understand", Leanne says she "usually likes knowing", and Betty says she has "to have meaning" if she is to understand something and have it with her for "life". This desire for what Cindy, Leanne and Betty perceive as an understanding of calculus distinguishes them from the other Collector students. This distinction will be discussed at a later point in this section, in conjunction with the other features that distinguish Cindy, Leanne and Betty from the other Collector students.

However, from the discussion thus far it can be concluded that Collector students' externally oriented *sources of conviction* do not promote a sense of personal understanding or ownership of calculus skills and conceptualizations. In responding to calculus problems, particularly visually oriented mathematical representations such as graphs, Collector students often use their language knowledge. However, they do not necessarily acknowledge the role their language knowledge plays in the construction of calculus conceptualizations. Although Collector students exhibit *sources of conviction* that are due to personal language knowledge, they do not necessarily credit personally generated calculus interpretations. In fact, four of the Collector students explicitly devalued what they they perceived to be personal ways of interpreting or solving calculus problems. The four students, Cindy, Doug, Ned and Daniel said such things as:

> (Cindy)
> . . . because I'm not perfect at interpreting it in the correct mathematical language.
>
> Well I might be able to write it down, but it probably wouldn't be right. I probably wouldn't do it the correct way. But I would, if I was to go back and read it I would understand what I meant. But it wouldn't be the right way so anybody else would understand it.

> (Doug)
> It just doesn't come to me easily. So I have to really work at it. Whereas something like English I can just do it. Political science. . . . There I can actually use, like I can just do it with my own mind. I can give my own interpretations of something. But in math it's either right or wrong.

> (Ned)
> . . . like with me I have to look through someone else's eyes. Like a foreign kind of viewpoint. That's very hard for me to do.

> (Daniel)
> I don't know if root is the right word because probably mathematically speaking that's probably wrong.
>
> Well he's a math professor. And I'm a political science major, university student. And I think the reason he's in math and I'm not, like that in itself is the fact that we both understand things differently. . . . and the way I describe things is more along a different line. You know, than mathematical notation. So if I told him an example about apples or pencils, he'd just kind of go X. You know. It's just, like to him it's not what he wants you to know. And I have a hard time grasping. That's probably why I'm failing. . . . Oh, it's definitely different. But it's not, it's not, it's not mathematical. I mean it's just not. . . . So that's not acceptable.

These excerpts demonstrate that Cindy, Doug, Ned and Daniel view mathematics as a "foreign kind of viewpoint" that must be interpreted in "correct mathematical language". They do not see as valid their personal ways of interpreting or expressing mathematics. Thus, Cindy, Doug, Ned and Daniel do not allow internal *sources of conviction* to play a prominent role in the building of their calculus conceptualizations. The other Collector students did not as explicitly devalue what they saw as personal ways of construing calculus. However, all the Collector students, except Leanne, spoke of calculus as being separated from their reality. They saw calculus as useless, or they perceived calculus to be different from other subjects or previously learned mathematics. For example, Collector students spoke of their impressions of calculus as follows:

> (Daniel)
> So to me I don't think calculus is incredibly useful. But I'm not the one to judge whether it is.
>
> Understanding. Well it means two things I guess. In math it means being able to produce it on the test. And understanding to me in the broader sense, English or whatever, means kind of capiching (sic) what they're saying about something. And being able to apply this knowledge. But I've never been able to do that in math. . . . Because it's above and beyond what it seems like I can comprehend.

> (Ned)
> But calculus seemed a little mysterious.
>
> Well calculus I would figure would be, he says to get lot of uncommon things. And some things are so uncommon they are never actually defined.

(Doug)
Like I've done good in math all along. Like on my departmental for Math 30 I got 88, 89. I come here and do math and it's not even like math.

Understanding in calculus is being able to do it. . . . Well, I just plain don't understand calculus. Like I get the questions, but it's not because I understand them. It's because I just memorized them. I don't know how to do it. But in other subjects, like anything, you read and you understand it. Like if you get an economics concept, you just understand it.

(Gordon)
In calculus. Same in physics. If you get the right answer you understand it.

If I read something once in history it's there forever. But in calculus it just seems to be there for about thirty seconds and then it's gone. . . . I think because I don't find it interesting. I don't see the point. Like history happened. It was real people. But this is just numbers on a page.

(Ellen)
It's just stuff. And you go like why am I doing this. Like if I majored in calculus, like what would I do with it?

Well what is calculus about? It's sort of weird. I don't know. I don't really understand it. I don't understand why you do those things. Like I don't know. . . . But calculus is a lot of letters and stuff, and I don't understand why you are doing certain things. Like it doesn't seem like there is a reason. Like if you were out in the world and you had to use your math that you learned in calculus, I don't know what you'd use it for.

(Cindy)
Calculus. That it's difficult. . . . Well it's different from a lot of other maths though, isn't it?

I mean normally in other math, like in the story of my math I have been. I've wanted to take the time to do things and figure out different things on my own. But I just found in this class I just got so frustrated. . . . Maybe I can't understand it the way I want to understand it. Like maybe, you know, like maybe because so much of it is abstract things, that you just have to be satisfied with not. I don't know.

(Betty)
It's a new type of math. From what we've learned all our lives.

The above set of excerpts indicate Betty, Cindy and Leanne do not as strongly as the other Collector students see calculus as separated from their reality. Leanne's words are not present in the excerpts because she did not speak of calculus as useless or different from other mathematics. Cindy expresses frustration with calculus because she has not been able "to do things and figure out different things" on her own. She is frustrated because she recognizes she has not been able to understand calculus in a way she would like. Finally, although Betty says calculus is a "new type of math from what" she has learned previously, she makes it clear in later comments that she sees calculus as a subject that can be understandable and meaningful to her. Examples of what Leanne, Cindy and Betty said that set them apart from other Collector students are the following:

> (Cindy)
> And I try to figure out the answers. I try to understand it from what he's written out. I mean just giving me a blank answer isn't the only thing I need. I need the whole solution to a problem to help me out. Like so I can know whether or not I did it, you know, whether or not I did it right.
>
> I'm learning by rules way more than I like to. Like I don't like to learn that way.

> (Leanne)
> I've been doing it like if the teacher says so I'll believe it. But if I had more time I would have to be, I have to, to feel comfortable in a course I want to know how I get the answer. Why I get it this way, and I want it proven to me.

> (Betty)
> Because, like if the learning is there you'll know it the rest of your life type thing. But if you do it robotic, like the last two chapters I did, I know I'm not going to do that well on the test on those two chapters. Because I don't know what's behind the derivative.
>
> And that's how it connects to, like behind the derivative, it's like the rate of change of a function. And I know that, and it's connected to the derivative. Now I remember the derivative. I know what it really means. And I can be more at ease in getting the answers. If you have an answer, you know what it means. If you just try and do it, what's a derivative, you're just sort of unsure of what it is. And the answer doesn't mean anything to you. When you do have the understanding behind it then it means a lot more. And then you understand it more. And that's how it connects.

Leanne, Cindy and Betty's words reveal how they each differ from the other Collector students as to what they perceive might be possible in their calculus learning. Each of them perceives the possibility that calculus can be personally understood. That is, they are able to perceive of calculus as a body of knowledge in which things "connect" and in which one knows how to "get the answer" without blindly following what the teacher said. Thus, although Leanne, Cindy and Betty are functioning as Collectors and have *sources of conviction* similar in nature and role to the other Collector students, they would choose to function differently if circumstances were different. In other comments they make it clear their adoption of a "memorization" approach to calculus learning is due to external constraints related to time and the amount of material covered in calculus. For example, they said the following about the limitations they found themselves under: (Leanne and Cindy are from Beta College (concepts-first instruction), while Betty is from Gamma College (infinitesimal instruction)).

> (Leanne)
> Like he'll ask are there any questions. And there's so many that you just can't get it into one question.
>
> In my high school classes I had a chance to, he gave the students a time where they could work out a problem. And then he'd do it, after a chance to figure it out. And then you could check whether you did it right or wrong. Whereas here it's just, he does it for you and it may or may not be what you would originally have done.
>
> (Cindy)
> Oh, I would like to be able to understand it. But it would take time. A lot more time than I have to put into it.
>
> But now he teaches it all in one lesson and we go home and we do the assignment, we find out there's problems there that he didn't either emphasize or not, or that we didn't realize there's problems. Cause when he explains it, of course we basically understand it. Until you go home and do it yourself. But then he's already taught it and he's on to something else. And you don't want to intrude and get him to do that problem the next class.
>
> I think it would be really beneficial if I had the time to sit down. If I had two or three hours everyday after class to go home and read through it and study it. I know that I could be way better at understanding it.
>
> (Betty)
> Because I stopped working at the understanding. I did the homework and everything, but I didn't look at it deeper.
>
> Some of it was time constraints. It was easy for me at first because there was rarely anything else to do. And I knew calculus was a hard course, so I thought maybe I should spend more time on it. Then I started getting behind in my other courses. So that was when it was harder to keep up.
>
> But when I am pressed for time robotic comes in and just starts going. Cause I can't waste any time on trying to learn it.

In these extracts Cindy and Betty talk of external time constraints as an important factor in their calculus learning. Betty comments that her "robotic" mode of functioning "comes in and just starts going" whenever she is pressed for time. She makes a distinction between this mode of functioning and a mode where one looks at material "deeper" and works at understanding. Cindy's comments are similar to Betty's. Cindy speaks of a desire to understand calculus, but finds this is inhibited by a lack of study time. She also expresses a frustration that her calculus course does not allow time in class to resolve difficulties before going "on to something else". Leanne also expresses a sense of frustration that the amount of material presented in one calculus class does not allow her time to work and practice with calculus ideas and problems. Thus, it is clear that under fewer time constraints Leanne, Cindy and Betty feel they would be able to learn calculus differently than they are at present.

In comparison, the other five Collector students did not explicitly express an awareness or desire that it be possible or important for them to acquire personal understanding of calculus. Their comments on their calculus learning included the following:

> (Doug)
> I'm kind of a pacifist about not understanding. If I don't understand it I don't understand it. As long as I get the marks. . . . It's not like I'm a math major where I have to understand.
>
> Like I said, if I know it good enough to get it right on the test, that's all I worry about.

> (Daniel)
> And right now it is a matter of being able to produce it on a test. And whether or not my interpretation is correct doesn't matter. Because my interpretation isn't going to be counted on the test.
>
> Well in most anything else I could feel confident my views are um maybe not necessarily correct, but that they're feasible, or that I can show how my views and somebody else's views correlate or something. Like you know. In math I don't feel that I have got any basis to say that I'm right and I'm wrong. Because if they, they referring to math people, come up with all this stuff, or how do I say it. I'm just not confident that my way of viewing it, like I could so easily be wrong. Like I just don't feel I have it.

> (Ned)
> I know I'm going to be paying an engineer to be doing all the difficult math. Like reconfirming like the techniques and structures that I would design. . . . So that's the extent of what I feel I need to know.

> (Ellen)
> I: Is it okay with you that you feel you memorize a lot?
>
> E: Well it is because if I just want to get through this course I don't really care. I'm not taking math again, that's for sure. Unless I really have to.

> (Gordon)
> For something like English or history, when you understand something you have to understand why someone did something, or the situation, or the events. But when you do calculus it's just numbers and you have to reason out those numbers. I don't think it's as relevant to life as something like English is. To me the interest isn't there, so if I get the right answers it's fine.
>
> But this is sort of just my course that if I pass it, great. If I don't, I've always got next year.
>
> G: I'm relying on being told that this is how everything works.
>
> I: Is that okay with you?
>
> G: It is in this course. If I was going on it wouldn't be. But this is just something I need, so I just do it. Accept it and leave it.

The words of these Collector students, Doug, Daniel, Ned, Ellen and Gordon, reveal they have virtually abandoned any efforts to personally understand calculus. Instead, they have accepted getting problems "right on the test" as a prime objective of their calculus learning. They are satisfied if they "get the right answers" and get "the marks".

At this point it must be noted that Daniel's feeling of a lack of confidence in his calculus abilities was not an isolated feeling amongst Collector students. All the Collector students except Betty and Ned explicitly expressed a lack of confidence in their abilities to personally understand calculus. Their comments in relation to a lack of confidence included the following:

> (Daniel)
> I don't have a lot of confidence in my own work because I know I'm not good at it. And so I doubt my abilities to get it right.
>
> I'm just not confident that my way of viewing it, like it could so easily be wrong. Like I just don't feel I have it.
>
> (Gordon)
> I'm not very confident. I go into every test thinking I'm going to fail.
>
> I just get intimidated before I start doing it.
>
> (Ellen)
> No. I'm not confident. . . . Because I don't really understand it.

> (Doug)
> I'm still apprehensive. I don't want to take it. I'm unsure about it and it's just not something I like to do a lot.
>
> Cause it's more, all I'm doing is memorizing his examples. I'm not really understanding what's going on. I just don't feel confident. I feel confident in memorizing it.
>
> (Leanne)
> I've been doing it like if the teacher says so I'll believe it. But if I had more time it would have to be, I have to, to feel comfortable in a course I want to know how I get the answer, why I get it this way, and I want it proven to me. But so far it's not that way.
>
> (Cindy)
> But at the same time I think, well even as confident as I can be that I have it right it could still end up being wrong. Cause it's happened to me so many times before. And so I just begin to have no confidence in what I'm doing. . . . And so you just begin to lose confidence in your own confidence after awhile.
>
> Because I don't even understand what's he's trying to teach me. Why would I want to go and figure out something on my own? Like I just have no confidence in myself.

These Collector students speak of not being confident with calculus. Cindy says she has "no confidence" in what she does in calculus, Daniel doubts his abilities to "get it right", Gordon says he feels "intimidated" by calculus, Ellen states she is not confident because she feels she does not understand calculus, and Leanne makes it clear she is not comfortable with learning calculus without personal understanding. Doug also comments he does not understand "what's going on", and he concurrently says he does not feel confident. What is not clear in these students' comments is whether the predominantly external nature of their *sources of conviction* is related to a lack of confidence in calculus. However, it will be seen in the upcoming sections of this chapter that Technician and Connector students do not display the lack of confidence reflected in Collector students' comments. This fact would therefore seem to indicate a relationship between external *sources of conviction* and a lack of confidence in doing calculus.

In summary, Collector students generally display *sources of conviction* that are external in nature. Their *sources of conviction* reside predominantly in statements, rules and procedures presented by a teacher or textbook. The role of these *sources of conviction* is as a validation to the student that his or her calculus statements and problem solutions will be recognized as valid or correct by mathematicians or mathematics teachers. By way of this role Collector students' calculus conceptualizations are constructed as an assemblage or collection of relatively unconnected mathematical statements, rules and procedures. The unconnected nature of this collection is evidenced by low Completion Scores relative to other interview students.

Collector students generally believe mathematics has definite rules and procedures, and is a dichotomy of right and wrong problem solutions. They also view calculus as different in nature to other mathematics or other subject areas. In relation to this point, Collector students generally speak of mathematics as separate from their own reality, and they sometimes explicitly devalue their personal interpretations of calculus. In addition, most of the Collector students explicitly expressed a lack of confidence in their calculus abilities.

Three of the Collector students, although the nature and role of their ***sources of conviction*** were similar to the others, expressed more of a desire that their calculus learning be personally understood. The other five Collector students stated that, because they saw calculus as neither useful nor meaningful to them, they were satisfied with getting correct answers. Collector students make use of ***everyday language*** as a ***source of conviction***. They use ***everyday language*** knowledge as a ***source of conviction***, but do not display a sense of personal understanding or ownership of their calculus conceptualizations. Instead, Collector students speak of their calculus learning as "memorization" of calculus statements, rules and procedures.

### Technicians

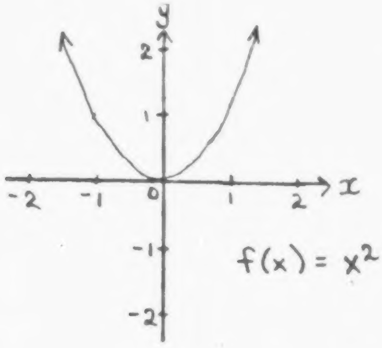
The students classified as Technicians are Jennifer, Richard, Sally and Nadine (see Table 9). Jennifer and Richard are from Alpha University, Sally is from Beta College, and Nadine is from Gamma College. These four students ranked eighth, second, fifth and thirteenth according to their Completion Scores (Table 8). These students display a mixture of internal and external ***sources of conviction***. Their external ***sources of conviction*** are similar to Collectors' in that they are based on knowledge of calculus statements, rules and procedures. However, Technician students differ from Collectors in their perception and use of these statements, rules and procedures. Technicians see calculus as a logical organization of statements, rules and procedures and they employ this organization as a technique for thinking about and applying calculus concepts. What therefore most distinguishes Technicians from Collectors is that Technician students display personal knowledge of how calculus statements, rules and procedures fit together into a logical whole. This logical whole thereby becomes a calculus "technology" in that it is a science or method for thinking about and applying calculus. Technician students can therefore be viewed as skilled users of the application of calculus techniques. Thus, the role of a

Technician's *sources of conviction* is as a set of tools that the technician employs to apply calculus concepts.

As already mentioned, Technicians display *sources of conviction* that are similar to Collectors' in their origins as knowledge of statements, rules and procedures. For example, Technicians said such things as:

> (Richard)
> Cause that's the way I was told to do it. . . . That's what I do. I learned that rule. I don't know why. I don't understand what a limit is. I just know how to find it.
>
> But I don't understand what I'm doing. I just know how to do it.
>
> I'm just trying, I'm just using the examples that he uses in his notes to make sure I'm doing it properly.
>
> (Sally)
> Well using the rules that we did it would be like subbing in that. But then you just take the highest power.
>
> I don't know. In class he told us when it's zero over zero it means more work.
>
> But then in math I work with rules. And how things work and what he says I just take as how you do it, you know.
>
> (Nadine)
> Because we always learned you take the derivative of the first times the second, plus the derivative of the second times the first.
>
> When you, the way we were taught is you take the point and you blow it up an infinite amount. And if you see a straight line then there's a derivative.

(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]



$$\frac{F(x+dx)^2 - x^2}{dx}$$

$$\frac{\cancel{x^2} + 2xdx + dx^2 - \cancel{x^2}}{dx} \quad 2x + dx \rightsquigarrow 2x$$

**Figure 31. Nadine's Written Response to Problem 5**

I: I've asked for continuity.
N: Yeah, but we were always told that using this shows any change in x. . . . I just know this, if I'm not mistaken that's the way we were taught to show it will exist at any point.

(Jennifer)
It's just, well for certain things. There's just, like there's certain rules which are just rules. And you can just build from there. But I wouldn't know how to get those basic rules.

Well we were always taught that if it's continuous you don't have to lift the pencil from the page.

The statements of the above interview extracts are reminiscent of Collectors' external *sources of conviction* in that they are based on what the teacher or textbook presented. For example, Richard says he "was told to do it" [a calculus problem] by use of a certain rule, Sally refers to what the teacher told the students "in class" about what a particular expression means, and Nadine and Jennifer refer to what they "were always taught" about interpretation of particular mathematical situations.

However, although Technician students make use of calculus statements, rules and procedures as external *sources of conviction*, a complete examination of their interviews reveals their calculus conceptualizations are more organized than Collectors'. Their knowledge of calculus statements, rules and procedures is more than the "collections" displayed by Collector students. More specifically, instead of a collection of relatively unconnected mathematical statements, rules and procedures, Technicians' *sources of conviction* are based upon statements, rules and procedures organized into a coherent, structured set. This set is then employed as a logical technique to think about and apply calculus concepts. The existence of structured calculus conceptualizations and related *sources of conviction* rather than an unorganized "collection" is partially evidenced by Technician students' Completion Scores (see Table 8). Their Completion Scores ranged from 22 to 31. Thus, three of the four Technicians had Completion Scores higher than seven of the eight Collectors. This fact indicates Technician students generally displayed more extensive calculus knowledge and skills. In addition, the organized structure of their calculus conceptualizations and related *sources of conviction* are displayed in the following interview extracts:

> (Jennifer)
> (Problem 6)
> [6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]
>
> The derivative. I don't know. It seems like it is a base that you can work from. And everything seems to rely on it. Like plugging values back into it. The points of inflection and critical points and stuff like that.
>
> (Personal interview)
> Understand something? To take that tiny basis of logic and be able to build on it. Like using that maybe as a cornerstone. But if you understand that, then you can understand things more. . . . Then you can continue onto a higher level. . . . By applying to another concept. How can I say it? Through practical application. I would understand derivatives by maybe drawing the graph or something like that. And knowing it would be right.
>
> I think calculus, if you get into a method of thinking it's just a process. It seems to be the same sort of process and you just get into that method of thinking and it's all very logical.

(Nadine)
It clicks. . . . When I did the derivative there, it clicked. You can do it, you can do it almost without thinking. You just know. You just know how to do it. You know a way to do it. You just do it. And you know your answer is going to be right or almost right. . . . Well, to be able to do that you have to understand it.

Well, if you know, if you know how to go. Like you know how to do the problem, but you don't understand how you did it. If you get a problem that isn't so clear-cut and you can't do it that definite way, and you have to understand how the problem works. Then, you know, the question can't be done. I relate this to chemistry because it's kind of the same thing. If you know how to find a concentration, okay. You just know how to do it but you don't know why you're doing it. You get to a question that asks you, it's just a little bit different, and you have to understand why your concentration is what it is. You can't do that clear-cut path anymore and you can't answer the question.

(Sally)
I'm trying to get the ideas into my head, and how to use those ideas. Like the examples especially. I follow them closely, step by step, what he's doing, how he's applying the ideas to a problem. And then I try to do them myself. And the exercises from the book.

I see what you're asking me as I do it. Like more, just the ideas of calculus. Like continuity. But then in class, then you have the equations and methods of calculation, and all that. . . . It's putting the ideas to use.

(Richard)
(Problem 4)

[4. What can you say about the function $y = \dfrac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?]

Because it was approaching zero in the denominator. . . . That told me it's going to have to be factored. Just from what I've done. All the exercises in the past. It means that it's going to have to be factored. If it approaches zero on the bottom. On no exam and during no assignment do they just have a real number on top over zero. Which would just be undefined. They never have exercises like that. The only exercises are, I just look at the denominator. If it's approaching zero then I'm going to have to do something with the top. It depends on the denominator.

(Personal interview)
I can see what's happening. Yes. I can see the pattern. But that's just because I've done it so many times that I can just see the pattern every time I just look at it.

Well, I have to visualize each step, before I do it. Otherwise I don't know what I'm doing. . . . I knew there was a connection, but then I just couldn't visualize it. And that's what I need to be able to do.

In these four sets of interview extracts Jennifer, Nadine, Sally and Richard speak of calculus as a building and application of a structure of statements, rules and procedures. Each student speaks of knowing how "to do" problems, or knowing how to "apply" calculus ideas to solve a problem. It is this sense of knowledge of how calculus is structured as an applicable technology that most distinguishes Technicians from Collectors. For example, Jennifer is a Technician rather than a Collector in that she sees calculus ideas and procedures as "a base that you can work from", "build from", and employ as a "process" or "method of thinking" to "work through" and solve problems. For Nadine, calculus "clicks" as a technique by which one can "understand how" and "know how to do" problems that don't have a "definite" or "clear-cut path" to an answer. Thus, one can say Nadine and Jennifer are Technicians in that they employ calculus as a set of tools by which to solve problems. Similarly, since Sally and Richard apply calculus as a technique for problem solving, they can also be said to be Technicians. Sally speaks of calculus "equations" and "methods of calculation" as "step by step" means of "putting the [calculus] ideas to use". Richard also speaks of calculus as a connected progression of steps. He refers to calculus as application of a "pattern" of steps that he must visualize in order to "be able to do" problems. These perceptions of calculus as a "method of thinking", a "pattern", or a logical "step by step" problem solving process were not present in Collector student interviews.

Thus, a prominent aspect of Technician students' *sources of conviction* is they are based upon knowledge of calculus as an applicable process or technology for solving problems. What is not clear in the analysis thus far is whether or not *sources of conviction* that reside in the technology of calculus are internal or external in nature. They would appear to a certain extent to be internal in that Technician students display a sense of personal control of how to apply calculus ideas and techniques. The students are more successful than Collectors in explaining and applying calculus, and they frequently outline what one does to solve calculus problems. In comparison, Collector students could not generally justify their work, except by reference to rules they stated they did not personally understand. The sense of personal control of calculus ideas that Technicians displayed was not present in Collector students' interviews. However, this fact does not necessarily imply Technician students' *sources of conviction* are internal in nature. When one examines comments on Technician students' own calculus knowledge, a mixture of external and internal *sources of conviction* appear. In this mixture it was often difficult to determine whether or not the students perceived their calculus knowledge as personally understandable. For each of the four Technician students this aspect of their *sources of conviction* will now be discussed. Their impressions of and experiences in calculus will be examined. First, Jennifer said such things as:

(Jennifer)
I: How do you decide when things are right or wrong?
J: Through logic, I guess. . . . But I would say that I know what I'm doing because the steps you take to achieve this answer are just so straightforward.

Usually it's like I use an example as just like a supplementary thing. Like what do I do next. Well okay, well let's see. So I can go from there if I get stuck.

This will sound kind of weird, but I find calculus it's just sort of a way of thinking. Then if you can establish that sort of process, then things just seem to make sense.

J: I think if you start understanding the theorems and how they derive these theorems, then that's understanding.
I: Do you feel you've achieved that?
J: To some extent. There's certain ones, you know, these are rules and okay, that's great, I'll follow these rules. . . . Well it would be nice to know what's going on. But sure, like it doesn't bother me too much.

The first three of the above interview extracts show Jennifer both sees and applies calculus knowledge that she views as personally understandable. It is personally understandable to her in that "through logic" she can perform "the steps you take to achieve" an answer to a problem. Jennifer also refers to using examples as a "supplementary" guide for her thinking, and she speaks of calculus as "things" that "make sense". However, the fourth extract reveals that concurrently with seeing calculus as a "way of thinking" that makes sense, Jennifer uses rules of which she does not necessarily feel a sense of personal understanding. She will "follow these rules" even though she doesn't necessarily "know what's going on" when they are followed. That is, Jennifer's *sources of conviction* are a mixture of internal, logical processes and external, rule-governed procedures.

Nadine's *sources of conviction* are similar to Jennifer's in their nature as a mixture of personally understood thought processes, and externally oriented rules and procedures. For example, Nadine said such things as:

(Nadine)
(Problem 5)
(see page 168 for Nadine's written response to Problem 5)
I just know this. If I'm not mistaken that's the way we were taught to show it will exist at any point.

(Problem 7)
It's not, it's just the way I do it. Cause I've done, this is one part I'm very good at, so I just do it.

If I can finish the problem to an answer that looks decent and looks somewhat to what I'm supposed to have, then I feel confident in it. . . . If it looks reasonable to what it should be.

> N: I try to understand it the best I can because my memory is not always that great. . . . But I try to understand it the best I can.
> I: Do you feel you are able to do that or not?
> N: Most of it, yeah. A basic grasp of it.

Similarly to Jennifer, Nadine's interview excerpts display more than one type of *source of conviction.* Nadine speaks of trying "to understand it [calculus] the best" she can, and she states she feels she has "a basic grasp" of an understanding of calculus. She speaks of a personal sense of confidence in her calculus work, and refers to when an answer "looks reasonable". Thus, Nadine's words reflect *sources of conviction* that are internal in nature in that they are perceived by Nadine to be personally understandable. However, Nadine also talks of her problem responses in terms of "I just know this" and "just the way I do it". In particular, she is not able to justify her responses to Problems 5 and 7, except as procedures that she has been taught. Since she is able to "just do" these procedures, but has no way to explain them, it appears she relies on these procedures as an external *source of conviction.* The overall picture of Nadine's *sources of conviction* is therefore as a mixture of external and internal sources. Her internal *sources of conviction* reside in personal confidence of an ability to know what to do in calculus to obtain reasonable problem responses, while her external *sources of conviction* are based on knowledge of what she was taught "to do" to employ particular procedures.

A prominent feature of Richard's interview was his proficiency with calculus. A Completion Score of 31 ranked him second amongst the seventeen interview students. His interview responses revealed he had knowledge of calculus concepts, and skill at applying these concepts. That is, Richard was highly skilled as a Technician of calculus technology. He was able to successfully complete the clinical problems, while simultaneously explaining what he was doing and why he was doing it. This feature is seen in the following extracts from Richard's responses to Problems 3a, 4, and 6:

(Richard)
(Problem 3a)
[3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}]$$

$$= \lim_{x \to \infty} \frac{x + \frac{4}{x^3}}{1 - \frac{1}{x^2} + \frac{5}{x^3}} = x = \infty$$

**Figure 32. Richard's Written Response to Problem 3a**

x to the third down to here. And as you do that, then of course going back to the previous question, as x goes to infinity of any number over x, that number will approach zero. And so it's just x plus zero, over one minus zero plus zero. So it's just x over one, or just x. And as x approaches infinity it's just infinity.

(Problem 4)

[4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?]

See, if it approaches a number on top then it's just an undefined. Because, say this approached, if you plug in negative two and it reached negative one on top, over zero. You know that the function's just going to be undefined. And you're finished. And you can go away. Or if it approaches zero on the top but there's a number on the bottom. Say if it was negative one on the bottom and it approached zero on the top, then I know it would just be zero. But I know immediately by looking at the denominator that this fraction is going to have to be factored.

(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

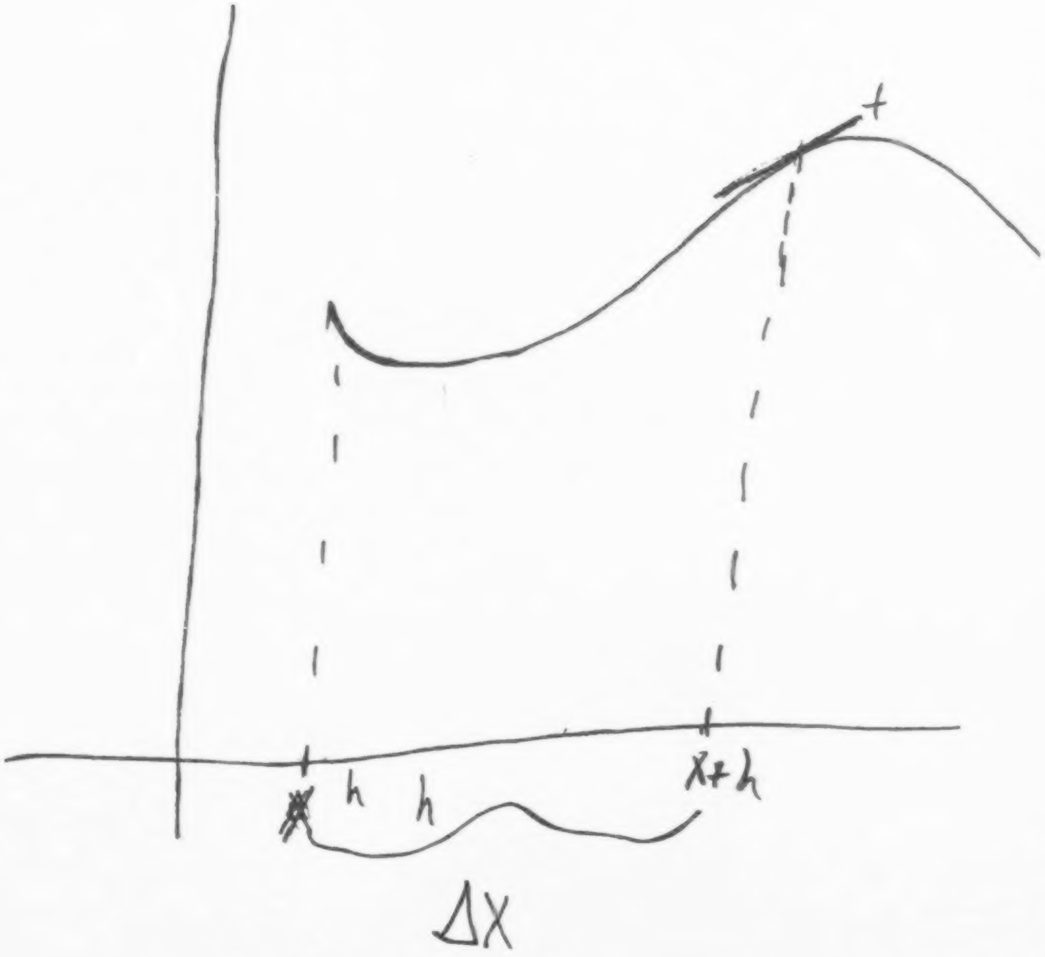


Figure 33. Richard's Written Response to Problem 6

> It's the slope of the tangent line right here at any given point. This is, well let's say, let me get another graph going there. So you're going to go. This is x. This is x. And then the rate of change. So here's your function. Let's just draw the function. And this is, ah, you want this point. And this point. That's x. And this would be x plus h. That's the point and that's x plus h. And that's the difference. So the difference here is delta x. And so what you can just plug in is you just, what you're trying to find is just the rate of change between this point and this point. The rate of change. What's happening on the function. You're just trying to discover what's happening on the function.

In these problem responses it is clear Richard has a grasp of calculus concepts and techniques. He displays consistent knowledge of concepts and techniques, and is able to explain his thought processes. These aspects of his interview, along with the fact he speaks of "visualizing" calculus steps or patterns (see page 170) indicate Richard's *sources of conviction* are to a certain extent internal. However, the extent to which they are internal is not clear. Richard's comments on his learning reveal that he does not always attribute his calculus knowledge and skills to his own abilities to make sense of and apply calculus ideas. His comments on his calculus learning included the following:

> (Richard)
> I can't prove it. I can't prove most theorems. I just memorize how to do them. I don't know how to prove them. . . . I memorize how to actually do, go about, you know, taking the theorem and applying it. That's what I memorize.
>
> I'm just trying, I'm just using the examples that he uses in his notes to make sure that I'm doing it properly.
>
> Well on the midterm I enjoyed that feeling of sitting down and knowing what I was doing was right. Not having any qualms. Not having any problems. Not having any doubts. Cause I knew what I was doing was right.
>
> Well I try not to make it just copying. You could do it that way. You could just open the book and just plug in the numbers for the different things. I try not to do that. So I try to sit there and close the book, and then do the problem myself.
>
> It's just experience. . . . The exposure has everything to do with it. If you're having trouble with graphing then you've just got to go home and do some more graphing.
>
> You get used to doing something. How does practice get you to any point? How do you become a better swimmer? You build on the expertise that you already have.

Richard speaks of memorizing how to apply theorems. He also speaks of using the teacher's examples as a guide for the proper way to do things. Thus, his words reflect external *sources of conviction.* At the same time Richard says his calculus studying is more than "just copying". He speaks of "doing the problem myself", and explains how personal "experience", "exposure", and "practice" help him build calculus "expertise". These latter comments indicate Richard's calculus learning is partially constructed from internal *sources of conviction*. Thus, as with Jennifer and Nadine, Richard displays a mixture of internal and external *sources of conviction*. His external *sources of conviction* are memorized theorems and application procedures, while his internal *sources of conviction* reside in making sense of his personal experience and practice with calculus. He actively involves himself in practicing and gaining experience with calculus skills.

Sally is similar to Richard in that she speaks both of memorizing rules and working through calculus problems for herself. For example, her interview included the following comments on her own learning:

> (Sally)
> Understanding is applying the ideas to get a right answer. . . . And you need to know the ideas in order to apply them. And know what ideas apply in what circumstances.
>
> When I do a question and I look at the book and the answers match. I guess that the whole way of understanding calculus for me is getting it right. But I guess for a lot of people, you know, it shouldn't be that. It should be just knowing the ideas. But for me it isn't.
>
> I'm trying to get the ideas into my head and how to use those ideas. Like the examples especially. I follow them closely, step by step, what he's doing, how he's applying the ideas to a problem. And then I try to do them myself. And the exercises from the book. And then also, I guess it's just memorizing rules. Though that's not what math should be. But it is a bit because you have to remember the rules in order to use them.
>
> I'm very stubborn, so I'll keep at it. I'll keep at the same problem for an hour.
>
> I: Do you feel a need to convince yourself?
> S: Yeah. Which is why I study. Why I do questions and assignments. To make sure.

Sally speaks of her calculus learning in terms that reflect both external and internal *sources of conviction*. The external nature of her *sources of conviction* are seen in her references to "just memorizing rules" and getting answers that "match" the textbook answers. Her internal *sources of conviction* are seen in the fact she tries to do problems and exercises for herself, keeping at "the same problem for an hour" and trying to learn "how to use" the ideas. Sally speaks of calculus as learning "ideas in order to apply them". To her, calculus is a technology of knowing "what ideas apply in what circumstances". Although she does problems for herself to "make sure" she is convinced of the validity or correctness of her work, she sees calculus understanding as getting the right answers. In other words, Sally's *sources of conviction* simultaneously reside in externally oriented statements of the textbook and personal, internal knowledge of the application of ideas. Thus, as with Jennifer, Nadine, and Richard, Sally exhibits a mixture of external and internal *sources of conviction.*

The external and internal mix of the nature of Technician's *sources of conviction* reveals their *sources of conviction* play a role in their learning by serving as a technology by which to apply calculus. Technician's calculus conceptualizations are thereby built and organized as a logical structure of appropriate application of calculus ideas and techniques. The external component of the related *sources of conviction* resides in externally generated statements, rules and procedures, while a personal sense of mastery of the rules and logical procedures of calculus gives rise to internal *sources of conviction.*

Technicians' personal sense of mastery of the technology of calculus indicates they see knowledge of the logical use of calculus language as a *source of conviction.* Through a coming to know how to use calculus symbols and terminology they organize their calculus experiences and structure their related conceptualizations. This language knowledge as a *source of conviction* was also evident in Collector students' interviews, but for Collector students it was primarily based in pre-calculus language knowledge. Technicians also displayed pre-calculus language knowledge as a *source of conviction* upon which to build calculus conceptualizations and this feature of Technician students is one way Technician students' characteristics can be viewed to encompass those of Collectors. Some examples of Technicians' use of pre-calculus language knowledge as a *source of conviction* are the following:

> (Sally)
> (Problem 3b)
> [(b) What does "limit" mean to you?]
>
> Something that a number approaches, but will never reach. Or something it can't cross like a border. Like you can't ever quite get to it.
>
> (Nadine)
> (Problem 3b)
> [(b) What does "round off" mean to you?]
>
> You don't have so many numbers to deal with. Cause, like with that one, if you have nine nine nine nine, you can continuously go on. If you have four, it's finite. It stops. And that is much easier to work with than three point nine repeating.

> (Jennifer)
> (Problem 9)
> (see page 64 for the graph for Problem 9)
> Well the function has, it's changing there [at x = -5]. Um. It really can't continue there, so it has to change there. So there can't be, it has to change, the function has to change at that point there cause it can't continue with the way the function was. So it has to make a drastic change. . . . If the function is unable to continue like that then I'd say the derivative wouldn't be able to.

In the first extract above Sally constructs a conceptualization of limit as a "border" that you "can't cross". Her previous knowledge of the word "border" and her association of the word "border" with the word "limit" have served as *sources of conviction* upon which to build her conceptualization of limit. Similarly, Nadine's previous knowledge of the term "round off" and her association of the term with the phrases "it stops" and it is "much easier to work with" have served as a base from which to build her calculus conceptualizations related to rounding off. Jennifer also makes use of previous language knowledge as a *source of conviction*. Her notions of "continuing" as not changing are carried over to her calculus conceptualizations of the behaviour of a function. A more complete discussion of language as a *source of conviction* is given in a later section of this chapter.

Another similarity between Technician and Collector students' interviews was that Technicians generally spoke of calculus as useless and different from other mathematics or other subjects. For example, Technicians spoke of their impressions of calculus as follows:

> (Richard)
> Well I just mean that in calculus when I say I understand something it means I can do it. And I can get the right answer. Whereas if I'm usually talking about another subject, well I understand the theory, or I understand the principles behind it. I know what is happening and I could, if somebody asked me to explain it to them, I could explain it to them in terms they could understand. Whereas I couldn't do that at all. I could never explain calculus to somebody in terms that they could understand. Because I don't understand. I just know how to do.
>
> I'm saying that the principles of calculus are useful to business, but coming in here and taking this calculus course is not useful to somebody who is in business.

> (Jennifer)
> There's like calculus mode and there's a math mode type thing. And calculus is different from other math.

> (Sally)
> Not only more difficult, but totally different. . . . It's a totally different kind of math. It's not, yeah, again, it's not dealing with numbers. And his examples, they're not examples. I don't know, they're not from life. You can't find problems that relate to life. They're just questions with a few numbers and some letters and words.
>
> That's another thing about calculus that's different from high school. In high school you do use it. But in calculus I just can't think of any real world illustrations of calculus.
>
> Just the fact that you can't see it in real life. It doesn't seems real. It's just talk. You can prove it, but that's just talk too.

In these interview extracts Jennifer and Sally speak of calculus as "different" from other mathematics. Sally also comments on how she sees calculus as "just talk" that is "not from life" and does not have "real world illustrations". Richard does not say calculus lacks real world application, but he does state he sees his calculus course as irrelevant and of no use to his future business career. It is noteworthy that, in spite of the fact they display mastery of calculus as a technology, Sally and Richard do not see calculus as personally relevant. This point further portrays the nature of Technician students' convictions as a combination of external and internal sources.

In summary, Technician students generally display a combination of external and internal *sources of conviction*. Their external *sources of conviction* reside in knowledge of calculus statements, rules and procedures, while their internal *sources of conviction* arise from knowledge of how to use these rules and procedures. Thus, the role of a Technician student's *sources of conviction* is as a means to organize and structure calculus statements, rules and procedures. The resultant structures thereby become a calculus technology that guides and informs the students in the application of calculus ideas and techniques. It is through a sense of mastery of the technology of calculus that Technicians' *sources of conviction* are more internal in nature than Collectors'. Their mastery of the technology of calculus is seen both in their relatively high Completion Scores (between 22 and 31 out of 36; see Table 8), and their comments on their own learning in calculus.

As with Collector students, Technicians use pre-calculus language knowledge as an internal *source of conviction*. However, Technician students do not display the lack of a sense of personal understanding of calculus conceptualizations that is present in Collector students' interviews. Instead, they use both their pre-calculus language knowledge and their newly acquired knowledge of calculus statements, rules and procedures to construct problem responses. Technicians expressed views that calculus is different from other mathematics or other subject areas, but they also spoke of an ability to solve calculus problems for themselves. Finally, although Technician students displayed some sense of personal mastery of calculus, they did not necessarily see calculus as personally useful or relevant.

**Connectors**

The students classified as Connectors are Annabel, Tim, Mike, Neil and Tanya (see Table 9). Annabel is from Alpha University, Tim is from Beta College, and Mike, Neil and Tanya are from Gamma College. These five students ranked fifth, fourth, second, fifth and first according to their Completion Scores (Table 8). A student who from his or her *sources of conviction* is classified as a Connector displays *sources of conviction* that are generally internal in nature. They are internal in that a Connector student displays a sense of being able to interpret calculus for herself or himself. Similarly to Technicians, Connectors display knowledge of calculus as a technology. They organize their calculus experiences so as to be able to logically and and consistently apply calculus ideas and techniques. However, Connectors differ from Technicians in that they display a stronger sense of personal understanding of their calculus conceptualizations. They also display a higher degree of competence in both explanation and application of calculus. Their conceptualizations are displayed as a network of "connections" between various aspects of calculus, and between calculus and the student himself or herself. In this way the role of a Connector student's *sources of conviction* as a validation to the student that she or he makes statements, performs procedures or creates problem responses that are valid, correct and meaningful to the student as well as other individuals. Thus, Connector students are able to both apply calculus knowledge and make personal sense of this knowledge.

Connector students frequently spoke of understanding as important in their calculus learning. They also spoke of approaching their learning as trying to connect together ideas, statements, rules and procedures. Examples of what they said in relation to these two features are:

> (Annabel)
> I feel fairly confident. . . . Cause I make sure. I try to make sure I understand everything as I go through it.
>
> I'm following along in class. Lots of times when he does a problem I don't copy it down until after he completely finishes. And then I understand it. Then I copy it down.
>
> Practice helps you gain confidence I think. And you learn more about them [symbols]. And also in this course lots of things build on what you did before.
>
> I'm trying to fit it all together. Not memorizing.

(Mike)
I like to know why something works. . . . But ah, I do like to be told how things work and if it doesn't make sense I'm not really afraid to say hey, hold it type thing.

I like, like I said, I like to know how it works for myself. And figure things out for myself. You have more control that way.

The background behind it. If I can look at a question and do it that doesn't necessarily mean I understand it. Like taking derivatives. Ah. A lot of people, like I said, I was explaining to my friends. Some people can take the derivative of any function using the rules, and not even understand it. It's knowing the process behind it and how the whole thing is going to go. If you can do that, explain how that works, then you can say you've fully understood it.

(Tim)
Try to understand. And difficult, and the questions is, are quite difficult. . . . and develop a skill for the students to try to think.

Try to understand the theory. Try to understand how it works.

Knowing why is important. Yeah. Because knowing why and how it work, ah, how it works and you can just develop your skill to do any kind of question which is similar to the theorem.

(Neil)
Quite confident. . . . I'd say almost more confident than my other courses. Ah, I understand I guess. For the most part I understand it. Where it comes from and why it is.

I don't think I would understand half the amount I understand unless I knew where it came from to begin with.

I definitely try to recreate things and think it through.

I've had it both ways. I had it just here's this formula and this is how you use it. . . . And I did lousy. Because then comes the exam, and they're going to throw these really strange things at you to see if you understand it. And I didn't know where it came from. I didn't know how to utilize it. How to alter it to suit my, to suit the problem in order to solve the problem. But when you know where it comes from then you can, it's just not this formula all set in stone. And then it's useful that way.

And here, the way he's teaching it you can see the connection, and I guess that's important. Seeing how everything is linked together. And not just this idea, and this idea over here. And if they are connected then one should know it. Even if it's a little more complex. But, I think the connections are important.

> (Tanya)
> That it makes sense. That it makes sense. Yeah, okay, it makes sense.
>
> Because you can't learn from memorizing everything. Because you have to interpret it. You have to understand the theory behind a certain form. The theory behind a certain something, and then apply it to something else.
>
> Cause you need to, you need to imagine it in your head. What goes on. You can't, you can't see infinity. You have to imagine infinity. You can't see infinitely, or infinitesimally small. You have to imagine it.
>
> Well from unit to unit you fit together everything that you learned before. Like limits applies to hyperreals, and derivatives applies to hyperreals. Everything you learn applies to infinitesimals and infinites. It all fits together.

In these excerpts the students make it clear they feel they understand calculus. Annabel says she feels "fairly confident" with calculus because she makes sure she understands everything as she goes through it. Mike says he understands things in terms of knowing the "process behind it" and how things are "going to go". Similarly, Tim says he tries to understand "how it works", because he can then develop "skill to do any kind of question". Finally, Neil says he knows where things "came from to begin with", and Tanya says calculus "makes sense" to her. In comparison, none of the Collector students said calculus made sense to them. Technician students expressed some sense of personal understanding of calculus, but they did not speak of their learning of calculus in the same way as did Connectors. Connectors spoke more of calculus as something one learns through personal involvement with and subsequent flexible application of ideas. This aspect of their learning is particularly clear in Tanya's comments on her learning. She speaks of her "imagination" as an essential component of her calculus learning, and notes how she must "interpret" rather than memorize in order to learn how to apply calculus theory. Through these words Tanya expresses a sense of personal understanding or ownership of her calculus knowledge. That is, as sources of conviction she uses knowledge and thought processes that she conceives of as her own.

This sense of oneself and one's own thought processes and interaction with material as sources of conviction by which to learn and use calculus is also seen in the other Connector students' words. Neil speaks of trying to "recreate things" for himself and he emphasizes he sees it important to know how things are connected together and how one can "utilize" them. Tim speaks of understanding as development of a skill "to try and think". He also says "knowing why" is an important goal of his calculus learning because it allows him to apply calculus to "any kind of question". Mike also speaks of personal understanding as a goal of his calculus learning. He says he has "more control that way". Finally, Annabel says she approaches her calculus learning by "trying to fit it all together".

She says she doesn't just "copy" down whatever the teacher does, but rather, follows along and tries to understand it.

The sense of personal control and involvement of one's own thought processes that is demonstrated by Connector students reveals the internal nature of their *sources of conviction*. More specifically, Connector students' *sources of conviction* are internal in nature in that they reside in a sense of personal comprehension and control of calculus ideas and applications. In this way, the role of Connector students' *sources of conviction* is as both a guide and a confirmation for the student that she or he states and uses calculus ideas and applications in ways meaningful to herself or himself as well as others knowledgeable in calculus.

It must be noted at this point that Connector students' *sources of conviction* are not exclusively internal in nature. As already mentioned, the nature of Collectors', Technicians' and Connectors' *sources of conviction* can be viewed as lying nested one inside the other. Similar to Collectors and Technicians, components of Connectors' calculus conceptualizations appear to have been built from externally generated statements, rules and procedures. For example, Connector students spoke of some aspects of their calculus learning in the following ways:

> (Neil)
> Well when you have x and y terms together you have to differentiate in terms of x in order to find y prime. And so you have to treat this as a separate function and do the chain rule on it. You do the derivative of the outside and the derivative of the inside function. And I'm not exactly clear on how that relates. . . . But in terms of the time constraint I just, like I knew, I could see right away how it was arithmetically. But in terms of conceptually I didn't have it as clear as I wanted to have it. But I needed it learned for the exam. I needed it learned.

> (Mike)
> But if we need to be sure on how something works, then yes, I want to know how it works. And, but then I also look at it that it is, it is, almost anything, any course. Math specifically, like memory work. You don't really have to think when you're doing something like the multiplication table. You have to memorize that and be familiar. So it's not an application where you're throwing in some numbers. It's memory work. You know. Six times seven is forty-two.

> (Tanya)
> But sometimes it's going through it that bothers me. It exists. Ah. It wouldn't be right if it wasn't right. So let's just work with it. We don't need to know how some blowjoe came up with it.
>
> I'm satisfied with just knowing where they got the basics from. Um. The definition of the power rule I found most fascinating. . . . But now when you get to the higher, higher steps like this with the chain rule, I think okay, well, I've seen all the other proofs. I know it works. I'll just know this.

> (Tim)
> So most of the time, if I don't understand I just try to memorize.
>
> I don't have time. If I have time I will practice more and get much understanding.

> (Annabel)
> I think that if you sit there and question why you're doing everything in calculus you won't ever get anything done. So you just say ah well, they're teaching it to us for a reason. And don't worry about why. Just try and understand what they're doing, not why they're doing it.

The above interview extracts show that aspects of Connector students' *sources of conviction* are similar to Collectors' and Technicians' in that they reside in externally generated statements, rules and procedures. Further, these statements, rules and procedures are knowledge of which Connector students do not claim to have a complete or personal understanding. The students speak instead of portions of their calculus learning as "memory work" of things that need to be "learned for the exam". Neil comments on how the time constraints sometimes affect his learning by requiring he know certain things for an exam before he has time to be "exactly clear" on how things relate. Similarly, Tim notes how he will "just try to memorize" when he doesn't have time to work on understanding something. Tanya and Annabel also refer to constraints. They speak of how the amount of material that is covered and the need to use material requires they not always worry about "going through" where something came from, or question "why they're doing it". They therefore focus on "the basics", and understanding "what" is being done. That is, their learning is influenced by the fact they must get things done and must be able to "work with things".

Thus, it can be said that Connectors have external *sources of conviction* that reside in memorized, "just learned" calculus statements, rules and procedures. However, the role of these external *sources of conviction* is somewhat different than the role of Collector and Technician students' external *sources of conviction*. For Collectors and Technicians, external *sources of conviction* served as a validation to the student that he or she makes statements or performs procedures that are in accordance with what either the technology of calculus or mathematics teachers or textbooks dictate. Collectors do not claim to have the ability to personally understand these statements, rule and procedures, and Technicians see them as internal only in that they feel they have mastered the thinking processes that comprise calculus technology. In comparison, it appears that the role of Connector students' external *sources of conviction* are as "fillers" between the calculus ideas and techniques they have connected and construed for themselves. They are fillers in that Connector students do not necessarily speak of the knowledge they have built from external *sources of conviction* as knowledge of which they would not be able to make personal sense. Rather, it is knowledge they have built from external *sources of conviction* because the constraints of their world necessitate they know particular things within a specific time frame.

Another feature of Connector students' interviews that was both similar and different to Collectors' and Technicians' was their views of mathematics. Their views were similar in that Connectors spoke of mathematics as being definite and set in its ways, but they differed in that Connectors spoke more of mathematics as a practical or human endeavour. In relation to these features they said the following:

> (Mike)
> Sort of like most math things. That you, ah, that you're taught all along and told to, this is how God made things and it's what you do. A lot of it is, I think like anybody can get like really high marks in math if they just realize that it was more memory than applications. Because math is so set in its ways. Like I said before, you can do math one way. Like there may a number of different ways of getting around certain problems just by different ah theorems and stuff like that. But you're more or less doing the same thing.
>
> It's, it can't be interpreted differently by a number of people. Like say a poem can. Because you know the guy that came up with the first little bit of calculus wasn't just writing it to please himself or anything like that. Like a poet could be writing a poem because he, you know, liked the sight of a bird flying across the sky or something like that. It was done because it worked, and then it was expanded upon, and it was just very logical.

> (Tanya)
> The best way I can say it is you will write an essay and every time, no matter how perfect the essay is, somebody will find something wrong, or a different way to do it, or a different way to interpret it. Sentences can be infinitely, and paragraphs can be infinitely juggled around. And math can only be done one way. One or two ways. There's certain ways, you know. You take a step here, step two. Math can only be done one way, and that's the only way I'll have to worry about knowing it.
>
> Like just the power rule, simple power rule. Um. Product rule, quotient rule I find are great because I find that somebody didn't just wake up and write this down and I have to study it now. They actually did think it through. They actually did come up with it.
>
> (Neil)
> I think one important thing about calculus is it's taking things that are infinite in value and making them finite. . . . It's taking things that are almost impossible to manage to conceptualize and giving more concrete ideas to them.
>
> It's not just barfing up formulas and seeing how much you can remember. It's taxing your mental abilities to see do you know, can you understand where it comes from.
>
> (Annabel)
> Most of the examples we take are practical examples. But as to whether somebody would actually do them, I don't know. But I think they are, there are times when calculus is useful.

Another prominent feature of Connector students' interviews was they displayed a higher level of competence with calculus concepts and skills than the other interview students. The Completion Scores for the Connector students, Annabel, Tim, Mike, Neil and Tanya, were 29, 30, 31, 29 and 32, respectively (see Table 8). These scores were amongst the five highest for all the interview students and this higher degree of competence with calculus concepts and skills was reflected in the nature of Connector students' problem responses. However, although there is evidence of a relationship between high competency in calculus and approaching calculus learning as a Connector, it is not clear if one causes the other. Their problem responses were often more detailed, using more symbolic representations and more complete explanations of ideas or procedures. For example, their problem responses included:

(Mike)
(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

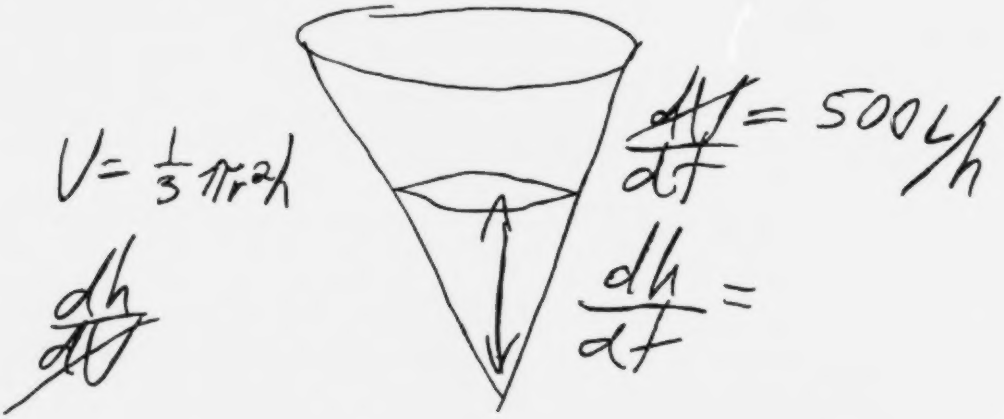


**Figure 34. Mike's Written Response to Problem 6**

Your volume filling. Like that would actually be given to you. In number of litres per hour or whatever. It's filling up. And then the question would ask you actually how fast is the height rising? So you need dh by dt. And ah, and then you have to figure out using the volume at this point. π r squared h or something. Something like that. Um. So using this, ah, if you could find a relationship somewhere in there with your r and h or whatever, and try to make them all equal. Then you could find your dh dv. You know. And multiply basically this times this. . . . And then you have your answer, how fast your height is rising. And when they ask me like where would that come in handy? . . . I told them open the bowl of the back of the toilet. . . . But you know that little switch? I don't know, maybe I've fixed our toilet at home too many times. And I'm starting to go funny or whatever. But you've got to, the chain comes down and hooks onto this little lever, right. And you get your plug. And on the back of it you've got this little tube with a hole in the back. Of course when your water's in there and you pull up. Like this thing is up now. And this thing starts draining all the water. Well you want to know really how fast it's going to go to give enough time for the bowl to drain. . . . I just thought that was kind of neat when I thought about it.

(Neil)
(Problem 8)
[8. What interpretations do you have for the expression below?

$$\frac{dy}{dx} = \frac{F(x+dx) - F(x)}{dx}]$$

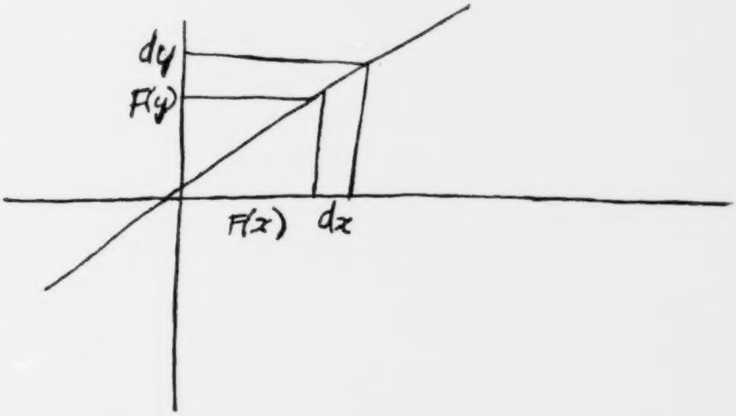
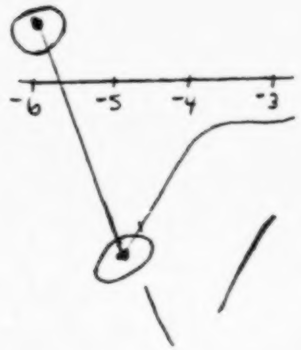


**Figure 35. Neil's Written Response to Problem 8**

When you're looking at a function magnified an infinite amount you have the value x and you want to know how it's changing. So if you go any amount over, well you go an infinitesimal amount over and you look at how the y value has changed with respect to how the x value has changed. And that will give you how the whole function is changing at that point. Because you're looking at it infinitesimally.

(Tim)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]



$$\lim_{x\to a^-} F(x) = \frac{f(x+h)-F(x)}{h} < 0$$

$$\lim_{x\to a^+} F(x) = \quad `` \quad > 0$$

$$\lim_{x\to a^-} \neq \lim_{x\to a^+}$$

**Figure 36. Tim's Written Response to Problem 9**

This point I consider does not have derivative [at x=-5]. Because derivative is the slope, right. The slope. They do not have the same slope I think in this graph. . . . From the left, from the left, from the right side this point we choose another point just close beside and we get the slope positive, right. And from the left side we choose another point close to this one. We get a negative slope.

(Annabel)
(Problem 3a)
[3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}]$$

$$\div \frac{x^3}{x^3} = \frac{\cancel{\lim_{x\to\infty}} x + \boxed{4/x^3}}{1 - \boxed{1/x}^2 + \boxed{5/x^3}} \lim_{x\to\infty} x = \infty$$

$$\frac{\infty^4 + 4}{\infty^3 - \infty + 5} = \infty$$

**Figure 37. Annabel's Written Response to Problem 3a**

I just divided top and bottom by the highest power of x on the bottom, x cubed. . . . Because then you can simplify the equation. You divide it by the highest power of x on the bottom because if you divided by the highest power of x on the top you'd have an undefined equation. Because it would end up being zero in the bottom if the, well because the top power is higher than the bottom power. And you do it to simplify it. To get rid of ah some of the things. Because you know it's the definition that something, one, any number divided by infinity will equal zero. Approach zero. As the limit approaches infinity. . . . Like you could go infinity to the fourth plus four. But I can't really see what you're accomplishing. That would be the simplest way. I guess you know that it's still going to be infinity because infinity to the fourth is obviously larger to infinity to the third minus infinity.

(Tanya)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

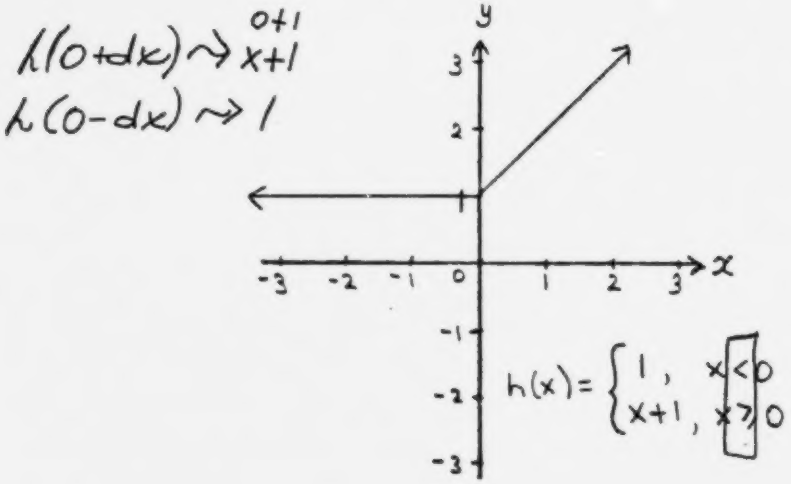

$h(0+dx) \rightsquigarrow \overset{0+1}{x+1}$
$h(0-dx) \rightsquigarrow 1$
y
3
2
x
-3
-2
-1
0
1
2
3
-1
-2
-3
$h(x)=\begin{cases}1, & x\boxed{<}0\\ x+1, & x\boxed{\geq}0\end{cases}$

**Figure 38. Tanya's Written Response to Problem 5**

> This is a function by itself. Even though it is a split function it's still a function. Okay. You can consider it as the same function. Because it connects right at zero. You can see by the signs. This is a less than and equal to. This is a less than. Here it connects with the same idea I was saying right here. Take an infinitesimal point right to the left. It will round off to the function itself at x. Or, yeah, okay. And the point right left of zero to the point, to the function, which is one.

In comparison to the problem responses given by Collector and Technician students, the above problem responses show more facility with calculus ideas and techniques. These problem responses also show how Connector students used language in the form of symbols as a *source of conviction*. That is, it is through symbolic language use that Connectors have constructed some of their calculus conceptualizations and related problem responses. Thus, as with Collectors and Technicians, language use serves as a *source of conviction*. Some of the Connector students, similarly to Collectors and Technicians, displayed everyday language use as a *source of conviction*. This aspect of *sources of conviction* is discussed in a later section of this chapter, as is students' use of symbols as a *source of conviction*. However, it must be noted that there is insufficient evidence at this point to determine whether symbolic language use as a *source of conviction* is internal or external in nature. Connector students demonstrate a familiarity and facility with symbols, but it is not presently clear whether symbolic language use as a *source of* *conviction* is perceived by them to be personally meaningful. Since this point relates directly to symbolic language use it is discussed further in a later section of this chapter.

In summary, Connector students generally display *sources of conviction* that are internal in nature. Their *sources of conviction* reside largely in ideas and techniques they perceive to make sense. That is, Connector students view calculus knowledge as something of which they can gain personal understanding and use. They speak of approaching their calculus learning in terms of aiming to understand, make sense of, and flexibly think through and apply ideas and techniques. In this way Connector students use their internal *sources of conviction* to construct calculus conceptualizations of which they feel personal understanding. The role of a Connector's *sources of conviction* is therefore as a guide and a confirmation to the student that he or she makes statements and performs procedures that are meaningful and useful to the student as well as other individuals. Connector students see their own interpretations and thought processes as components of their calculus learning. Their calculus conceptualizations are thereby constructed as a network of personally meaningful, interconnected statements, rules and procedures.

### *Language Use*

As with *sources of conviction*, during extensive examination of the student interview transcripts the context category for *language use* proved to be in need of change. This examination revealed that *language use* and *sources of conviction* are highly intertwined. In fact, it became clear that *language use* is a *source of conviction*. It will be demonstrated it is a *source of conviction* in that language knowledge serves as a foundation from which students construct their problem responses. The nature and role of students' *language use* and its relationship to *sources of conviction* will be discussed in upcoming sections. First, it must be pointed out that as *sources of conviction* categories were thrown into question, so too were language context categories. For example, in Sally's response to Problem 3a (see page 133) Sally apparently speaks both within mathematical and rule-oriented contexts. This feature in itself indicates that classification of the context of particular language statements cannot be uniquely determined.

To more appropriately and reliably analyze the context of a student's *language use* it was decided not to examine specific language statements. Instead, the context of *language use* was examined on the broader level of an entire problem response. This examination was done by noting if a student made use in a problem response of *technical language*, *everyday language*, visual mathematical representations or physical objects, none of which were supplied in the problem statement. These features, though not necessarily a reflection of a student's perceived context of his or her response, aided description of a student's *language use*. The details of the revised analysis procedures for *language use* will now be described, along with results of the analysis.

A student's *language use* for each clinical interview problem was determined by examining the related interview transcript and written responses. A Language List was made for each student for each problem. Samples of these Language Lists are in Appendix R. The lists are comprised of *technical* and *everyday language* words, phrases or expressions used by a student in a particular problem. Items are included in this list only if they were not already present in the statement of the problem. For example, a student's use of the word "derivative" would not be included on the student's Language List for Problem 6 because the *technical language* word "derivative" is present in the statement of that problem. Similarly, manipulations or operations with symbols present in the statement of a problem did not constitute inclusion of those symbols on a student's Language List. Thus, symbolic expressions were included on the Language List only if they constituted a symbolic representation not present in the statement of the problem.

Along with the Language Lists, a Language Chart was made for each student. These Language Charts are in Appendix S. The rows of each Language Chart correspond to the clinical interview problems. However, as has already been mentioned, Problem 1 responses were included with data related to a student's general comments on their calculus experiences. The columns of the Language Charts correspond to:

(1) symbolic *technical language* (TL-S),
(2) *technical language* words and phrases (nonsymbolic *technical language*) (TL-W),
(3) *everyday language* (EL),
(4) figures (F), and
(5) objects (O).

The columns labelled either "number" or "count" will be explained shortly. Figures include graphs, diagrams, or other visually oriented mathematical representations. Objects include reference to or use of physical entities such as a car, fencing around a field, or a swimming pool.

Although *technical language* has previously been defined to include mathematical symbols, the *technical language* category was divided into two subcategories at this point. Reasons for this division are: (1) the researcher became aware while conducting the interviews that many students made use of *technical language* words and phrases, but made relatively little use of symbols, and (2) students' use of symbols differed from their use of technical language words and phrases in that words and phrases were generally used orally, while symbols were written. Thus, for most students, use of symbols was distinctly different from use of nonsymbolic *technical language*. In fact, spoken *technical* and *everyday language* words and phrases were generally used as a main method of communication, while symbols were generally used for performing operations. Use of symbols to represent ideas occurred infrequently in comparison to use of nonsymbolic representations of ideas.

For each clinical problem, corresponding to a row of a Language Chart, a checkmark was made under each "number" column category present in the student's responses to the problem. The filled circles of the Outline Chart in Appendix S indicate which column categories were present in the written statement of each problem. For example, filled circles are under columns TL-S, TL-W and F for Problem 5 in the Outline Chart because the statement of Problem 5 includes equations for functions (TL-S), the word "continuous" (TL-W), and graphs (F). The circles provide easy reference to the explicit context of the problem statement. As with the Language Lists, checkmarks were made under a "number" column category variable only if the student made use of that category beyond what was already given in the statement of the problem. For example, a checkmark was made under the figure column (F) for Problem 5 only if the student introduced and made use of a figure (graph or diagram) not given in the statement of Problem 5. An example of this occurrence is the following graph Daniel drew and referred to it in his explanations for Problem 5:

(Daniel)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

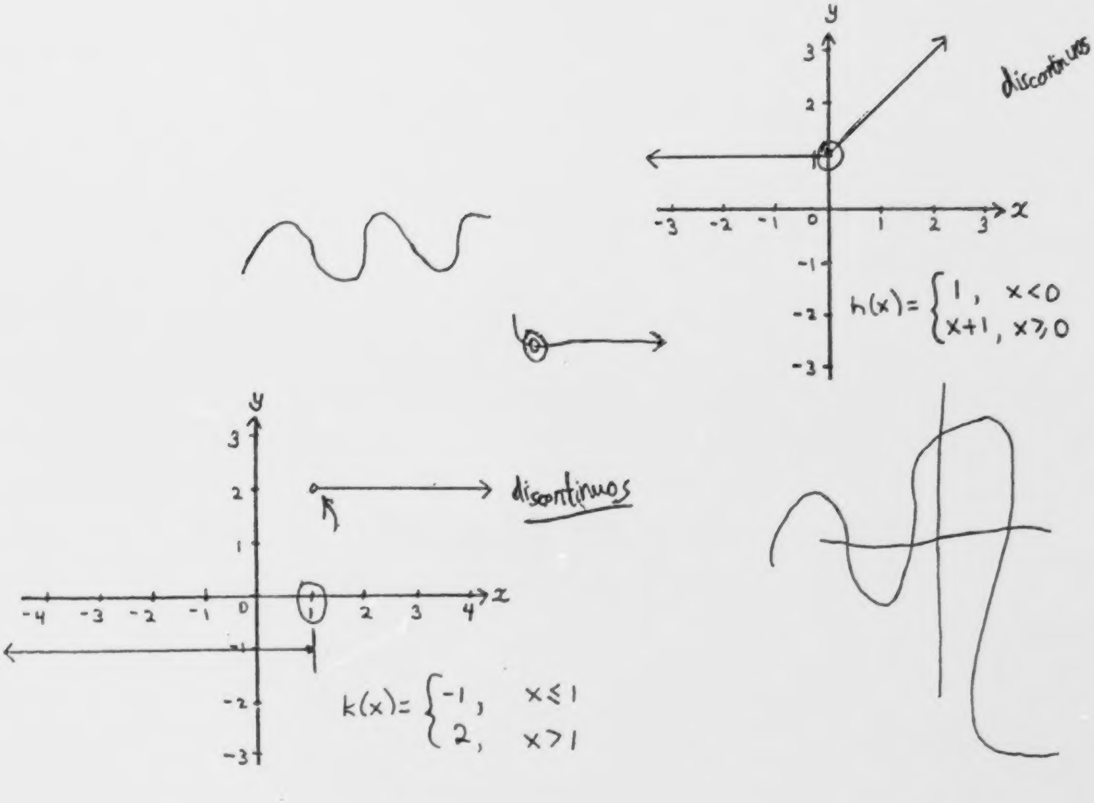
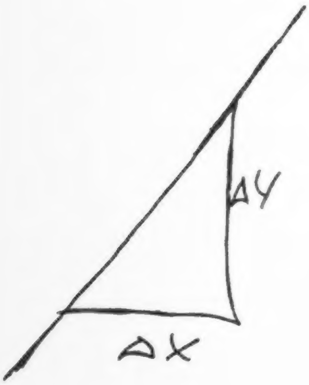

y
3
2
-3
-2
-1
0
1
2
3
x
-1
-2
-3
discontinuos
$h(x)=\begin{cases}1, & x<0\\ x+1, & x\geq 0\end{cases}$
y
3
2
1
-4
-3
-2
-1
0
1
2
3
4
x
-2
-3
discontinuos
$k(x)=\begin{cases}-1, & x\leq 1\\ 2, & x>1\end{cases}$

**Figure 39. Daniel's Written Response to Problem 5**

Similarly, a checkmark was made under the "number" column for symbolic *technical language* (TL-S) for Problem 7 only if the student's response to Problem 7 used symbols not already given in the statement of Problem 7. Use of symbols to label points, axes, or functions on a graph or diagram did not constitute use of new symbols unless the symbols themselves constituted a mathematical representation. For example, Tanya's written response to Problem 6 includes:

> (Tanya)
> (Problem 6)
> [6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]



$$y+dx-y$$

$$\frac{f(x+dx)-f(x)}{dx}$$

Figure 40. Tanya's Written Response to Problem 6

Here, the expression $\frac{f(x+dx) - f(x)}{dx}$ was considered use of symbols because it constitutes a mathematical representation not given in the problem statement. However, labelling of the sides of the triangle with $\Delta y$ and $\Delta x$ was not considered symbol use because the diagram was the main mathematical representation and the symbols were used as labels on the representation. Also, manipulations or operations with symbols already present in the problem did not constitute use of new symbols. For example, a checkmark was entered for Tim's response to Problem 7, but not for Annabel's response. Tim introduced operator notation as a representation of the differentiation process, while Annabel just performed operations with symbols. The relevant portions of Tim and Annabel's responses to Problem 7 are:

(Annabel)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7]$

$$F'(t) = 10(2t^2+3t-2)^9(4t+3)(3t^{1/4}-9)^7 +$$
$$(2t^2+3t-2)^{10}(7)(3t^{1/4}-9)^6(3/4t^{-3/4})$$

**Figure 41. Annabel's Written Response to Problem 7b**

(Tim)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7]$

$$= \frac{d(2t^2+3t-2)^{10}}{d(2t^2+3t-2)} \cdot \frac{d(2t^2+3t-2)}{d\textcircled{t}} + \frac{d(3t^{1/4}-9)^7}{d(3t^{1/4}-9)} \cdot \frac{d(3t^{1/4}-9)}{dt}$$

$$= 10(2t^2+3t-2)^9 \cdot (4t+3) + 7(3t^{1/4}-9) \cdot (\tfrac{3}{4}t^{\frac{-3}{4}})$$

**Figure 42. Tim's Written Response to Problem 7b**

In addition to checkmarks, numeric entries were made in the "count" language columns of the Language Charts (TL-S(count), TL-W(count), and EL(count) columns). The entries are the number of new language expressions, words or phrases the student made use of in each problem for each language column variable. For example, the number "2" was entered on Betty's Language Chart under the TL-W column alongside the checkmark for Problem 4 because Betty's response to Problem 4 included the *technical language* words "undefined" and "indeterminant" (see Appendices R and S for Betty's Language List and Language Chart).

The final row of each student's Language Chart gives the number of checkmarks in each "number" column and the sum of the numeric entries in each "count" column. They are referred to, respectively, as the total "number" and the total "count". Thus, EL(number) refers to the number of problems in which *everyday language* was used. EL(count) refers to the total number of *everyday language* words or phrases used in all the problems. Totals for each column were averaged across all students at each institution (see Table 10). For each student, Table 10 also contains the ratio EL(count) to TL-W(count).

That is, the ratio of the total number of *everyday language* words or phrases used to the total number of *technical language* words or phrases used. An average ratio for each institution is also given.

The choice of the column categories for the Language Chart was made so as to provide indication of the environment within which the student constructed his or her response to a problem. Totals for each column could then be used as indicators of the nature of a student's *language use*. Determination of the role *language use* played in interpretation of calculus problems came from examination of what students said or wrote and what this language reflected of their calculus conceptualizations.

Figures in Table 10 show both similarities and differences between the groups of students at the three post-secondary institutions. These similarities and differences will now be discussed, proceeding across the columns of the table. Values from the table are used as initial bases for the discussions. The discussions are then expanded to include findings from examination of what students said or wrote in the interviews, and what role these responses played in their interpretations of calculus problems. Throughout these discussions, unless stated otherwise, the term *technical language* will not include mathematical symbols. That is, symbols and purely verbal *technical language* will be discussed separately. The role of students' *language use* as a *source of conviction* will be discussed throughout the upcoming sections.

# Table 10. Column Totals and Institution Averages for Students' Language Charts

| Student/ Institution | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | EL (count) /TL-W (count) |
|---|---|---|---|---|---|---|---|---|---|
| Annabel / α | 6 | 7 | 11 | 23 | 9 | 14 | 3 | 1 | 0.61 |
| Ellen / α | 2 | 2 | 7 | 8 | 7 | 24 | 1 | 1 | 3.0 |
| Jennifer / α | 2 | 2 | 11 | 22 | 8 | 30 | 4 | 1 | 1.4 |
| Ned / α | 1 | 3 | 8 | 24 | 9 | 37 | 3 | 2 | 1.5 |
| Richard / α | 4 | 5 | 12 | 29 | 9 | 22 | 6 | 1 | 0.76 |
| **Alpha Averages** | 3.0 (25%) | 3.8 | 9.8 (82%) | 21.2 | 8.4 (70%) | 25.4 | 3.4 (28%) | 1.2 (10%) | **1.5** |
| Cindy / β | 3 | 4 | 11 | 26 | 10 | 25 | 7 | 2 | 0.96 |
| Daniel / β | 3 | 3 | 10 | 24 | 9 | 31 | 5 | 3 | 1.3 |
| Doug / β | 3 | 3 | 10 | 23 | 9 | 36 | 2 | 1 | 1.6 |
| Leanne / β | 2 | 4 | 12 | 30 | 10 | 25 | 8 | 0 | 0.83 |
| Sally / β | 5 | 5 | 10 | 30 | 11 | 32 | 6 | 3 | 1.1 |
| Tim / β | 7 | 7 | 10 | 30 | 9 | 16 | 7 | 0 | 0.53 |
| **Beta Averages** | 3.8 (32%) | 4.3 | 10.5 (88%) | 27.2 | 9.7 (81%) | 27.2 | 5.8 (48%) | 1.5 (13%) | **1.1** |
| Betty / γ | 5 | 8 | 9 | 16 | 11 | 43 | 4 | 1 | 2.7 |
| Gordon / γ | 1 | 1 | 9 | 18 | 8 | 29 | 5 | 1 | 1.6 |
| Mike / γ | 6 | 7 | 12 | 23 | 9 | 41 | 4 | 2 | 1.8 |
| *Nadine / γ | 6 | 6 | 8 | 18 | 10 | 41 | 2 | 1 | 2.3 |
| Neil / γ | 3 | 5 | 10 | 32 | 12 | 35 | 4 | 1 | 1.1 |
| Tanya / γ | 7 | 7 | 12 | 42 | 11 | 45 | 4 | 3 | 1.1 |
| **Gamma Averages** | 4.7 (39%) | 5.7 | 10.0 (83%) | 24.8 | 10.2 (85%) | 39 | 3.8 (32%) | 1.5 (13%) | **1.8** |

*The figures for Nadine are incomplete because there was insufficient time in her interview to do Problem 4. These figures are therefore lower than they would have been if she had done Problem 4.

### Symbols

At all three institutions students made relatively little use of symbols in comparison to their use of *technical* or *everyday language*. Considering the highly symbolic nature of calculus, this fact might be said to be surprising. Furthermore, since 76% of the Completion Scores (see Table 8) on the skill problems (Problems 3a, 7, 10) were 2 or 3 (partial or complete responses), it can be concluded that although students were able to perform standard symbolic operations, they did not generally use symbols to convey ideas.

The ease with which students often carried out symbolic operations but struggled with or were unable to provide symbolic mathematical representations can be seen in the following two excerpts from the interview with Jennifer:

(Jennifer)
(Problem 10)
[10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point $(0,-2)$.]

$$2xy + x^2y' + 2yy' - 3 = 0$$

$$x^2y' + 2yy' = 3 - 2xy$$

$$y'(x^2 + 2y) = 3 - 2xy$$

$$y' = \frac{3 - 2xy}{x^2 + 2y}$$

$$y' = \frac{3 - 2(0)(-2)}{(0)^2 + 2(-2)}$$

$$= \frac{3 - 0}{-4}$$

$$y' = \frac{-3}{4}$$

✓

Figure 43. Jennifer's Written Response to Problem 10

I did implicit differentiation. So using the product rule for the first term found the derivative of x times y. Plus the derivative of y times x. And then for the second term, implicit differentiation of x, y squared. Just differentiate that. Differentiation of a constant is zero. And then isolating y prime. Take all the terms which don't have a y prime to the other side of the equation. Take out the y prime. And divide the factored, what's left of the factored form through to the other side. You isolate y prime. Then just substitute the points in for x and y.

(Problem 8)
[8. What interpretations do you have for the expression below?

$$\lim_{h \to 0} = \frac{f(x+h) - f(x)}{h} ]$$

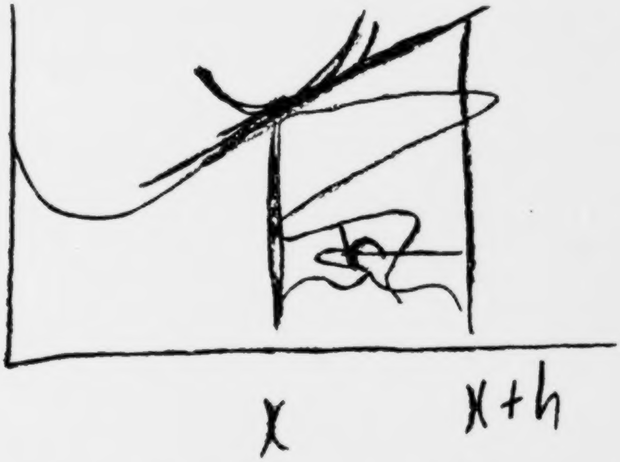


Figure 44. Jennifer's Written Response to Problem 8

J: That's the definition of the derivative using limits, isn't it? What interpretation would I have?

I: Do you have any way of saying more about it? Or showing where it comes from?

J: Um. Well, there's that graph. As well. Whatever. And this would be x. And then x, and x plus h. [long pause] Hm. h approaches zero. [long pause]

I: What are you thinking?

J: I'm not really sure. It would have to, I'm trying to think of, as x approaches, as h approaches zero. [pause] So is this, the distance between here. h approaches zero. The limit of that would just be the slope of that tangential line there. This would be getting closer to that. These lines would be there at that point f prime cause that's the definition of the derivative. It would be the tangential slope at that point.

In the first extract Jennifer proceeds quickly and accurately through implicit differentiation, demonstrating she has knowledge of how to work appropriately with symbols to differentiate implicitly. That is, *language use* in the form of symbolic procedures serves as a means by which Jennifer constructs her problem response. In the second extract Jennifer struggles with a graphical interpretation of the symbolic definition of the derivative. She knows the symbols are the definition of the derivative and that they relate to the slope of a tangent line. However, she is unable to provide an adequate explanation of the relationship between the symbols and the graph. She labels the points x and x+h on the x-axis, but does not identify the role of f(x) and f(x+h). She also does not explain how the symbols in the definition correspond to the slope of a line. Jennifer's Problem 8 response therefore shows how she has knowledge of the graphical location of x and x+h, but she does not have complete knowledge of how these symbols relate to use of the *technical language* terminology "derivative" and "slope of a tangent line".

Thus, it is seen that Jennifer has knowledge of appropriate symbolic procedures, but lacks knowledge of symbolic representations or interpretations. She clearly knows how to use a symbolic procedure as a technology for finding a derivative and the corresponding slope of a tangent line, and this procedure serves Jennifer as a *source of conviction* by which to construct her problem response. Whether it is an external or internal *source of conviction* is not however clear at this point. It must also be noted that symbolic *language use* in the form of symbolic representations does not appear to be a feature of Jennifer's *language use*. Thus, it appears symbolic *language use* in the form of symbolic representations does not serve Jennifer as a *source of conviction* by which to construct problem responses and related conceptualizations.

Jennifer was not alone in her display of skilled use of symbolic procedures and concurrent lack of ability in use of symbolic representations. From Table 10 it can be seen that 10 of the 17 interview students used symbolic representations in at most four (one third or less) of the calculus problems. Nine of these 10 students were Collectors or Technicians, while 4 of the remaining 7 students were Connectors. This fact indicates Collectors' and Technicians' *language use*, more so than Connectors', includes use of symbolic procedures but not use of symbolic representations. Another feature of the use of symbolic procedures was that Collectors often made errors in these procedures, but Technicians and Connectors generally performed them accurately. Examples of these features are the following (for further examples of Collector students' procedural errors see pages 151, 152 and 153):

(Leanne)
(Probiem 7b)
[$F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7$]

$$F'(t) = 10[2t^2+3t-2)^9, 7((3t^{1/4}-9)^6))$$

$$10(2t^2+3t-2)^9(4t+3) + 7(3t^{1/4}-9)^6\left(\frac{3}{4}t^{-3/4}\right)$$

Figure 45. Leanne's Written Response to Problem 7b

(Richard)
(Problem 10)
[10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point $(0,-2)$.]

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 3 = 0$$

$$y - y_0 = m(x_1 - x_0)$$

$$2xy - 3 = x^2\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$(y+2) = m(x-0)$$

$$\frac{dy}{dx} = \frac{2xy-3}{x^2+2y}$$

Figure 46. Richard's Written Response to Problem 10

(Mike)
(Problem 10)
[10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point (0,-2).]

$$2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 3 = 0$$

$$\frac{dy}{dx} = \frac{-2xy+3}{x^2+2y}$$

$$= \frac{-2(0)(-2)+3}{0^2+2(-2)}$$

$$= \frac{3}{-4} = -3/4$$

**Figure 47. Mike's Written Response to Problem 10**

In the first of the above three extracts Leanne (a Collector) uses the chain rule for differentiation, but uses it incorrectly in conjunction with the product rule. The second and third extracts show how Richard and Mike (a Technician and a Connector, respectively) are able to accurately proceed through an implicit differentiation. In all three extracts the students use knowledge of symbolic procedures, even if remembered inaccurately, to construct their problem responses. Thus, ***language use*** is the foundation upon which they build their problem responses.

Another feature of symbol use was that a number of students spoke of symbols as not having meaning for them. They said such things as:

> (Daniel)
> Because to me it looks like Greek on the board when he works through all that stuff. Well usually the notation doesn't make a whole lot of sense sometimes. To say that a represents a constant. . . . like derivative is equal to f at a minus, or f at b minus f at a, all over b minus a. To me, what does that represent? It's a little too ah, like to me it's easier if you just say that. Instead of writing it out with a's and b's. Like why don't they just say what they mean, you know?

> (Doug)
> Cause it's not black and white. It's not. Like stuff about a very small number and stuff. Epsilon, or whatever. It just makes it all the harder cause, I don't know why. It's not written right out, I guess.

> (Ellen)
> You have a variable x and y. Why do you have a d in front of it? Or why do you have a little slash thing on it having the derivative? Like what does that mean?

> (Cindy)
> I guess you have to have them but I just get really, really confused. There's so much. And I don't think there's enough attention given to making us understand all the symbols.

> (Leanne)
> It's mind boggling. With like epsilon, delta, and whatever else. Or epsilon one and epsilon two. It can get confusing to me because they're both little numbers. Like you see all those symbols and stuff, it kind of gets, you want him to just slow down and say what does this mean.

> (Richard)
> They're just symbols I move around according to a rule. They don't really mean anything.
>
> It doesn't have a meaning. It seems like it's stupid notation. Why don't they have notation that says what it is?

These extracts display how these students see symbols as separate from any personally understood calculus conceptualizations. They refer to symbols as things that do not "make a whole lot of sense" (Daniel), are "not black and white" (Doug), are "mind boggling" (Leanne), and make one "really confused" (Cindy). Doug, Daniel and Richard also comment on how they wish notation were more "written right out" so that it "says what it is". For these students, symbolic ***language use*** in the form of symbolic representations is not a ***source of conviction*** for the construction of calculus conceptualizations. Further, the fact that all these students except Richard are Collectors indicates there is likely a relationship between a lack of use of symbolic representations and external *sources of conviction*.

A relationship between symbolic *language use* and *sources of conviction* can also be seen in the problem responses of the 5 students who used symbols the most. These students are Annabel, Tim, Nadine, Mike and Tanya, who used symbols in 6 or 7 of their problem responses (see Table 10). Four of these five students are Connectors, with the additional student, Nadine, a Technician from Gamma College. What is noteworthy about the three students from Gamma College is that they, unlike most of the other students, gave symbolic justifications or explanations of continuity or differentiability in Problems 5 and 9. Furthermore, the symbols these students used and their corresponding verbal language were particular to the infinitesimal approach to instruction used at Gamma College. For example, Tanya, Mike and Nadine's responses to Problems 5 and 9 included the following:

> (Tanya)
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

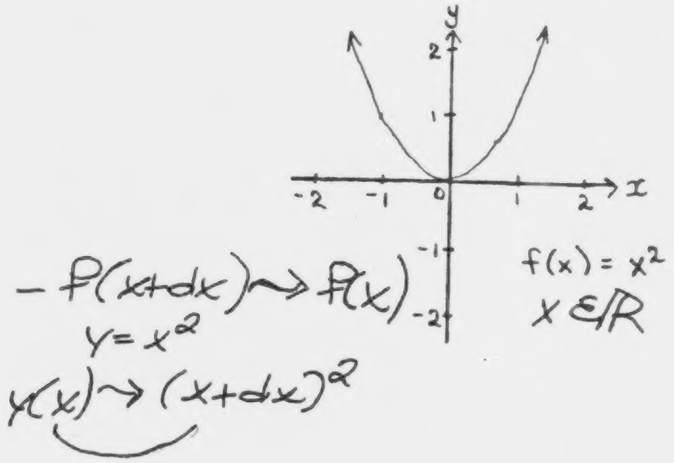

y
2
1
-2
-1
0
1
2
x
-1
-2
$f(x) = x^2$
$x \in \mathbb{R}$
$- f(x+dx) \rightsquigarrow f(x)$
$y = x^2$
$y(x) \rightsquigarrow (x+dx)^2$

**Figure 48. Tanya's Written Response to Problem 5**

> That any, I'll kind of do it this way. y at x. And these two x's are the same. Ah. If you take any x point and go a little bit to the left or a little bit to the right an infinitesimal amount, it rounds off to y at that x on the y-axis.

(Mike)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

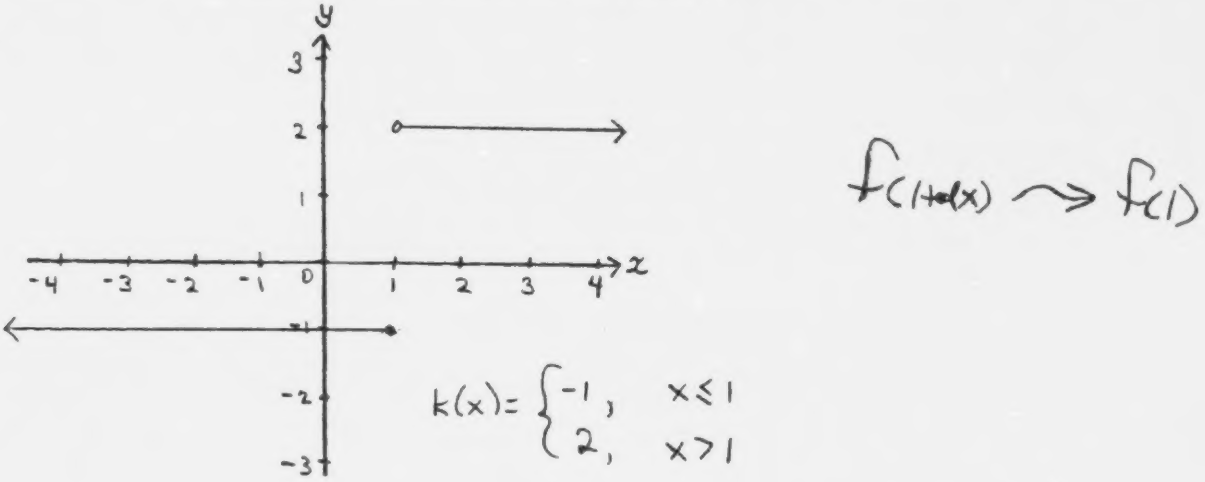


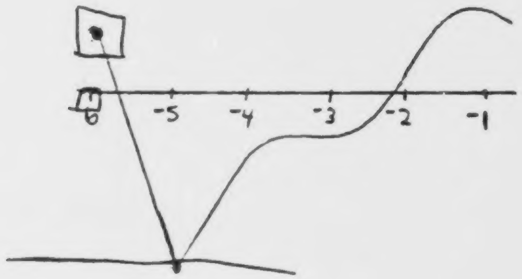
Figure 49. Mike's Written Response to Problem 5

M: So your dx could be positive or negative. Either side of it. But that would have to round off to the function at one. To be continuous. Now if you get a negative then it will. But if you get positive, all of a sudden you're going to . . .

I: If what's negative or positive?

M: Your dx. You know if you have negative dx you're going to be just to the left. And it's, well that rounds off to the correct thing. But just to the right of it, hey, I've got a two here, not a minus one.

(Nadine)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?]



$$\frac{f(-6+dx)-(-6)}{dx} = \frac{dx}{dx} \begin{matrix} \rightsquigarrow 0 \\ \rightsquigarrow 0 \end{matrix}$$

**Figure 50. Nadine's Written Response to Problem 9**

> N: Well if you want to find the derivative at a point, see I don't know what the function is. Okay, so I'm just going by f at x. And at your point you're looking at negative six. It's just a point. So you look at negative six and have negative six plus dx. And if there's any other point beyond there, that's what dx is. It's any infinitesimal x or change in x.
>
> I: So where would negative six plus dx be on this graph?
>
> N: To the left and right. Cause you have negative, cause you have the dx a little bit to the left, and dx a little bit to the right. The dx to the left does not exist.

In all three of these extracts the students use infinitesimal language (words and symbols both) to explain the relationship between the behaviour of a graph and the corresponding notions of continuity or differentiability. In doing so, the infinitesimal notation serves as a tool for construction of an explanation. It is a tool in that interpretation of dx as an infinitesimal number provides the students with something fairly concrete to work with. They easily visually locate on a graph what dx corresponds to, and how the position of dx relates to the behaviour of the graph. The students use infinitesimal notation as language by which to both build and justify their responses, and in this way infinitesimal *language use* serves these students as a *source of conviction*. However, it is not clear at this point whether this use of infinitesimal notation as a *source of conviction* is external or internal in nature.

Traditional limit notation can be used to explain and justify continuity and differentiability in ways similar to those of the previous three interview extracts. However, of the 11 students who received calculus instruction using the limit concept, only Annabel and Tim successfully justified continuity or differentiability by the use of limit notation. A third student, Jennifer, displayed a sense she knew limits were needed to prove discontinuity, but was unable to connect her ideas with the limit notation she wrote down. These findings suggest limit notation is not generally used by students as a *source of conviction* for construction of calculus conceptualizations. Annabel and Tim, the two students who were able to use limit notation to justify or explain concepts, were also the two students at their respective institutions who used symbols most frequently (see Table 10). They were also unique amongst students at Alpha University and Beta College in that they are the only Connector students from these institutions and the only individuals to display in their work and express in responses to personal interview questions that they appreciated, felt comfortable with, or had meaning for mathematical symbols. In relation to symbols they said such things as:

> (Annabel)
> They're just a good compact way of expressing it. It's more basic. It puts it all down in point form and it gets rid of the extra words. With the extra words, a lot of times it seems confusing. Kind of when you read Hamlet and you want to get rid of a few.
>
> I think they are important to my learning because it just makes it more basic and it makes it a lot less writing.
>
> (Tim)
> It means something to me. For instance, this thing is a small value [writes $\delta$ and $\varepsilon$]. I know this, I know take a derivative [writes $\frac{d}{dt}$].
>
> Math is international style. I think by the symbols.

In these extracts Annabel and Tim express how they see symbols as a means by which to express and think about mathematical ideas. Annabel sees symbols as important to her calculus learning because she can use them as a "compact", "more basic" means of communication. They do not have the "confusing", "extra words" that she sees as a part of verbal communication. Tim also speaks of how his calculus learning uses symbols that "mean something" and that are a means by which to "think". Thus, since Annabel and Tim perceive symbols to be personally understood, it can be said that symbol use serves Annabel and Tim as an internal *source of conviction.*

For the other three Connector students, Neil, Mike and Tanya, it has not been established whether symbol use as a *source of conviction* is external or internal in nature. Since they are Connectors it seems likely that their symbol use as a *source of conviction* is internal in nature. However, Nadine, a Technician, was also one of the students who used symbols most frequently. Whether or not these students' use of symbols as a *source of conviction* is external or internal in nature will be considered in the next section of this chapter, in conjunction with analysis of *technical* and *everyday language use*. It will be seen that Neil, Mike and Tanya's use of symbols differs from that of Annabel and Tim in that it is more highly intertwined with *everyday language use.* It will also be seen, in conjunction with other types of *language use*, that there is not enough evidence to determine if symbol use by Neil, Mike and Tanya (Connectors) is external or internal in nature.

In summary, symbols do not generally form a large component of students' *language use* in calculus. Although some students are able to manipulate or perform operations with symbols and thereby use symbolic procedures as *sources of conviction*, they do not use symbols to represent concepts. However, Connector students displayed more extensive use of symbols as *sources of conviction* by which to explain or justify ideas. The students from Gamma College did so by use of symbols and words particular to infinitesimal calculus. Infinitesimal symbols served them as objects that could be concretely represented on a graph and referred to in the construction of an explanation or justification.

### ***Technical*** **and** ***Everyday Language***

*Technical* and *everyday language* will be discussed in conjunction because it would be inappropriate to discuss one without reference to the other. This inappropriateness is because students generally used a combination of *technical* and *everyday language* in their responses to problems. Interview excerpts used to demonstrate the nature and role of students' *technical language* therefore often simultaneously display *everyday language use*, and vice versa. In addition, since *everyday language* is often given by students as an explanation of a concept denoted by a *technical language* term, it would be a misrepresentation of the students' *language use* to completely separate the two language types.

Students at all three institutions were similar in the extent to which they used *technical language*. The average use of *technical language* (TL-W(number)) at each institution for the set of interview problems ranged from 82% to 88% (see Table 10). The average total of *technical language* terms used (TL-W(count)) was also similar at all three institutions, ranging between 21 and 27. However, figures in Table 10 also show that, on average, students at Gamma College used *everyday language* in a higher percentage of problems (85% versus 81% and 70%). Though these figures are not in themselves significant, they are meaningful in conjunction with the figures for the totals for *everyday language* terms used (EL(count)), and the ratio EL(count) to TL-W(count). The average of the total EL(count) values at Gamma College was 39, whereas the corresponding averages at Alpha University and Beta College were 25 and 27, respectively. These figures show that, relative to the average total EL(count) values for Alpha University and Beta College, the average total EL(count) value for students at Gamma College represents use of at least 44% more *everyday language* terms.

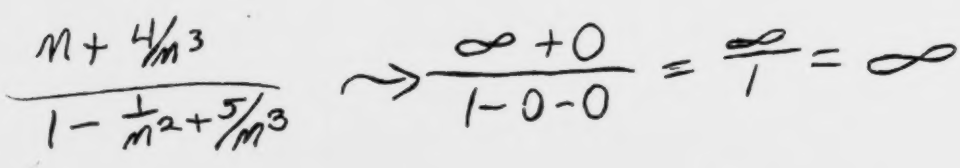
Students at Gamma College also differed from the other two groups of students in that all students at Gamma College had EL(count) to TL-W(count) ratios that were greater than one. In comparison, the same ratio for students at Alpha University and Beta College exhibited a greater variety of values. For some students the EL(count) to TL-W(count) ratio was greater than one, and for other students it was less than one. The two groups, students whose EL(count) to TL(count) was greater than one and students for whom this ratio was less than one, contained fairly equal numbers from each of these two institutions. Three students at Alpha University and three students at Beta College had EL(count) to TL-W(count) ratios greater than one, while two students at Alpha University and three students at Beta College had ratios less than one.

The TL-W(count), EL(count) and EL(count)/TL-W(count) values indicate that although students at Gamma College used *technical language* to about the same degree as students at the other two institutions, they used *everyday language* more. This finding distinguishes them from the other students. However, what distinguishes them more is the content of their *technical* and *everyday language use* and the ways they used related terms to describe, explain or justify calculus ideas. These features will now be discussed. Examples of what students at all three institutions said and wrote are given. The nature and role of language in interpretation of calculus problems are discussed, and in particular, *technical* and *everyday language use* as a *source of conviction* is discussed.

Students' problem responses revealed some important features of their *language use*, including: (1) conceptualizations built using infinitesimal language displayed features different from conceptualizations built using traditional calculus language, and (2) whether speaking with traditional or infinitesimal language, students used terminology as a *source of conviction* by which to construct conceptualizations, and in these constructions pre-calculus language knowledge was prominent.

To begin with, when asked what "round off" meant to them, students at Gamma College gave explanations not completely congruent with the corresponding responses given by students at Alpha University and Beta College to explain "limit". Gamma College students generally spoke of rounding off in terms of making numbers or calculations "less messy" or "easier to work with". They said such things as:

> (Mike)
> [(b) What does "round off" mean to you?]



$$\frac{m+4/m^3}{1-\frac{1}{m^2}+5/m^3} \rightsquigarrow \frac{\infty+0}{1-0-0} = \frac{\infty}{1} = \infty$$

**Figure 51. Mike's Written Response to Problem 3a**

> It's less messy. Ah. Like I'd really rather look at ah something like an answer here [his answer in Problem 3a]. If it's infinity. Than this whole mess [the original expression].

> (Tanya)
> It's better to work with it. It's better to work with that number, and for all practical purposes it is kind of really that number.

> (Betty)
> When you take like um, not a whole number, but like decimal numbers. And you try to get them to round off to whole numbers.

> (Gordon)
> It just makes it easier to work with numbers. If you have sixteen divided by three point nine nine nine, or sixteen divided by four. This [4] is a whole lot easier to work with than that is [3.999 . . .].

> (Nadine)
> It's making a problem a little bit simpler. . . . You don't have so many numbers to deal with. Cause like with that one, if you have nine nine nine nine, you can continuously go on. If you have four, it's finite. It stops. And this is much easier to work with than three point nine repeating.

An important feature of these excerpts is they reflect notions related to the round off process that students are taught in elementary school. That is, students appear to have built their conceptualizations of rounding off from pre-calculus mathematical experiences of the term "round off". They refer to making numbers simpler or easier to work with, and also speak of numerals after a decimal place as an important aspect of rounding off. The pre-calculus term "round off" and its related pre-calculus interpretations can therefore be said to serve students as a foundation for building conceptualizations related to the *technical language* term "round off". Further, since it is from this foundation that the students build their calculus conceptualizations, *language use* is seen to be a *source of conviction*.

However, previous familiarity with the term "round off" also appears to have prompted some students to retain round off notions that are not always congruent with the role rounding off plays in calculus. For example, Betty's belief that the result of rounding off must be a whole number is a misconception of rounding off in both a pre-calculus and calculus context. In addition, in the last two extracts above, the way Gordon and Nadine used the *everyday language* phrases "makes it easier" and "little bit simpler" does not necessarily reflect accurate notions of the role rounding off plays in calculus. Rounding off is a process that is generally done at the final step of a calculation. Thus, although it might make the resulting answer "a little bit simpler" or "easier to work with", it does not necessarily ease previous calculations or symbolic operations. It might be that Gamma College students' conceptualizations of the role of rounding off includes the notion that rounding off makes it "easy" or "simple" to work with calculus ideas, as well as simplifying answers. However, this possibility is not clear from the above excerpts.

A carryover of previous conceptualizations associated with terminology was also evident in the interview responses of students at Alpha University and Beta College. In these students' explanations of the term "limit" it was clear that students' interpretations of the calculus term "limit" contained a mixture of everyday and mathematical interpretations. Furthermore, it was clear that this mixture arose from students' conceptualizations of "limit" as an *everyday language* term. Examples from the interviews included:

> (Ellen)
> Well if it has a limit then, yeah, it's going to approach a number and then when it gets to that number it's going to stop. It's not going to get any bigger or smaller. . . . but it will always come back. Like it won't really go anywhere. Do you know what I mean?
>
> (Richard)
> It can't ever reach it. In other words it's just getting closer to it.

(Ned)
Well, limits just don't seem to be clear. But um, so you have to really assume all limits. Say if you're drinking you should know your limit. Well it's so uncertain, but you should, you should get a general idea of what your drinking limit is.

(Doug)
I usually take limits as x towards something. I just put it in there.

(Daniel)
The limit for myself represents a barrier or endpoint at which something is possible. For example, a swimmer would only be able to swim one mile because that is the limit of his or her endurance. Similarly in math, though more complex, a limit represents a maximum or minimum possibility.

(Sally)
Something that a number approaches, but it will never reach. Or something it can't cross, like a border. Like you can't ever quite get to it.

(Leanne)
You have a limit or a number that a certain equation or a certain curve is approaching, and it will never actually get there. But it won't go beyond there.

(Cindy)
C: A series of numbers that approach one number. That approach the same number.

I: Can you actually reach a limit?

C: No. You can only get close to it.

Before discussing these excerpts it must be noted that the two Connector students at Alpha University and Beta College, Annabel and Tim, did not display the misconceptions of limit that were displayed by Collector and Technician students at these institutions. The Collector and Technician students, by ascribing meanings to limits in terms of drinking, swimming endurance, stopping, getting closer but never reaching, going towards something, being a barrier or being something you can't cross over, revealed the powerful role previous language knowledge played in their construction of calculus conceptualizations of "limit". The excerpts reveal the *everyday language* term "limit" and the conceptualizations students relate to it served as a *source of conviction* for the students' constructions of conceptualizations related to the *technical language* term "limit". These everyday conceptualizations appear to have led students to construct narrow notions of limits.

However, since the limit concept is used in calculus to derive other calculus concepts, it is important to also examine how students made use of their limit-related language in problems requiring application or interpretation of limits. The same examination of ***language use*** must also be done for students at Gamma College and their use of infinitesimal language. It will be seen as these discussions proceed that ***everyday language use*** was both a help and hindrance for students' interpretations of calculus problems. Whether it was a help or a hindrance depended on the extent to which students integrated ***everyday language*** with ***technical language*** (including symbols) in ways congruent with the corresponding concepts.

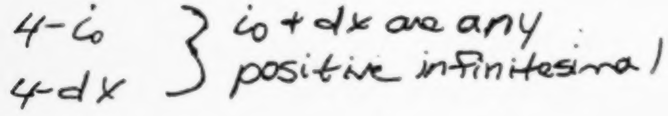
For example, students' Problem 2 responses generally appropriately integrated symbols and ***everyday language***, explaining the behaviour of the two sequences with ***everyday language*** phrases such as: "getting closer and closer", "getting smaller and smaller", "really really really really small", "very very close to", "so close", "gets close", "goes to", and "bigger and bigger and bigger". This use of ***everyday language*** aided students' interpretations in that, because the phrases were consistent with concepts underlying the ***technical language*** terms "limit" and "round off", it allowed them to reach correct conclusions. For example, "really small" and "very close to", though not necessarily complete interpretations, are correct interpretations of particular limit situations. In comparison, ***everyday language use*** in responding to Problem 2 hindered one student, Doug. He interpreted the symbolic language of the sequences in terms of the ***everyday language*** phrases "continuing on" and going "forever". These phrases are valid ***everyday language*** interpretations of the situation, but they led Doug to the mathematically incorrect conclusion that the sequences did not have limits. Thus, for Doug, inappropriate integration of symbols and ***everyday language*** resulted in construction of a mathematically incorrect calculus conceptualization.

Examples of appropriate language integration for Problem 2 between symbols and ***technical*** and ***everyday language*** was displayed by two students at Gamma College, Tanya and Nadine. Tanya and Nadine used ***technical language*** alongside ***everyday language*** in their response for the second sequence (3.9, 3.99, 3.999, . . .). They used ***technical language*** particular to infinitesimal calculus, and Tanya even gave her words a symbolic representation. Tanya and Nadine's responses to Problem 2 included:

(Tanya)
(Problem 2)
[2. For each of the following sequences of numbers, decide whether the sequence rounds off to a particular number. If so, what is this number?

$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$

3.9, 3.99, 3.999, 3.9999, 3.99999, 3.999999, . . .]



$\left.\begin{array}{l}4-i_0\\ 4-dx\end{array}\right\}$ $i_0$ + $dx$ are any positive infinitesimal

Figure 52. Tanya's Written Response to Problem 2

> And it's going to get closer and closer to four. . . . Four minus an infinitesimal amount. Which isn't four, but it's as close to four as you get.

(Nadine)
(Problem 2)
[2. For each of the following sequences of numbers, decide whether the sequence rounds off to a particular number. If so, what is this number?

$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$

3.9, 3.99, 3.999, 3.9999, 3.99999, 3.999999, . . .]

> Well this one, this one is gradually increasing towards four because you keep adding on another nine. . . . Well it's just when you keep adding on another nine you're increasing the value. And eventually you'll be adding an infinitesimal, an infinitesimal amount.

In these two excerpts Tanya and Nadine used the *technical language* term "infinitesimal" and the corresponding notion of an infinitesimal number to justify their previous explanations using the *everyday language* phrases "get closer and closer to four" and "gradually increasing towards four". For Tanya and Nadine, *technical language* arising from an infinitesimal approach to calculus was connected to symbolic and *everyday language use*. In addition, Tanya and Nadine's appropriate integration of *technical* infinitesimal language and *everyday language use* was not an isolated incident amongst students at Gamma College. As the interview transcripts were analyzed it became clear that Gamma College students' *language use*, in addition to inclusion of more *everyday language*, differed in its integration of *technical* and *everyday language*.

Students at Alpha University and Beta College did not display much integration of symbols, *technical* and *everyday language*. The two exceptions were Annabel and Tim, who were both Connectors, and were the two students at Alpha University and Beta College, respectively, who used symbols most. Annabel and Tim also stood out as distinct in that their EL(count) values were the lowest of these values amongst students at their institutions. The values were 14 and 16, respectively, whereas the corresponding institution averages were 25 and 27. Thus, the nature of Annabel and Tim's *language use* was different from their peers. They did not use as much *everyday language* and symbols were used more, although *technical language* was used to about the same extent. The other students often gave valid explanations of situations using *everyday language*, but did not as frequently use *technical language* or symbols for further, more detailed or precise justifications. In particular, unless specifically asked to do so, they did not make extensive use of language and ideas related to limits. There were occasions when they used *technical language* or symbols but were unable to explain the connections to *everyday language* explanations. Examples of these occurrences are given below. The students in the first two excerpts use a mixture of *everyday* and *technical language* to explain their ideas, but *everyday language* is more prominent and they do not connect the related ideas to *technical language*. In the last two excerpts the students are unable to explain their symbolic and *technical language use*.

> (Ellen)
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.] (See page 61 for the graphs for Problem 5)
>
> Well this one is for sure continuous because it doesn't have any breaks in the graph [graph 1]. And I think this one is continuous too because it just, it doesn't have any breaks in this [graph 2]. But it does because it's in two different sections. But I just think it's continuous. I don't know why. And this is for sure discontinuous because it has breaks [graph 4]. And this, the little circle is the point at which it discontinues. And then this one, ah, it's continuous because it's just one straight line [graph 3].

(Sally)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.] (See page 61 for the graphs for Problem 5)

S: This one is continuous because it never stops [first graph]. There's no gaps. And there's no, like you could just keep drawing it forever.

I: If I asked you to prove it in some way could you? Algebraically?

S: Um. Algebraically, no. But I could draw you a picture of a parabola.

(Leanne)
(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

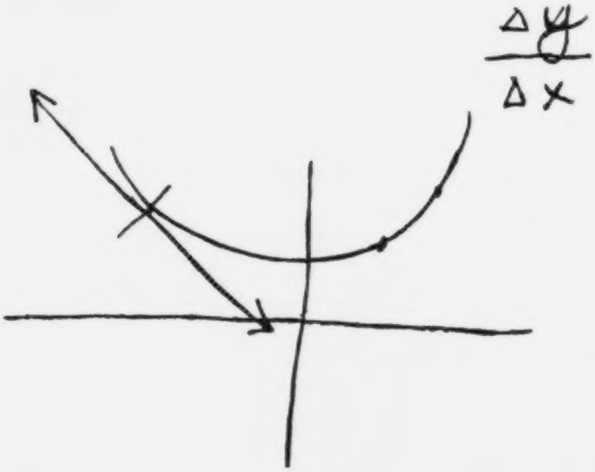


**Figure 53. Leanne's Written Response to Problem 6**

Like if you had a graph of a curve and you had a change in your x value and a change in your y value. It would be a change along the curve. And that change is represented by the derivative. And this is like the slope. It has something to do with that too. But I can't remember how it all relates.

(Jennifer)
(Problem 6)
[6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

$$f(x) = x^2 + x^3 + 5$$

$$f'(x) = 2x + 3x^2$$

**Figure 54. Jennifer's Written Response to Problem 6**

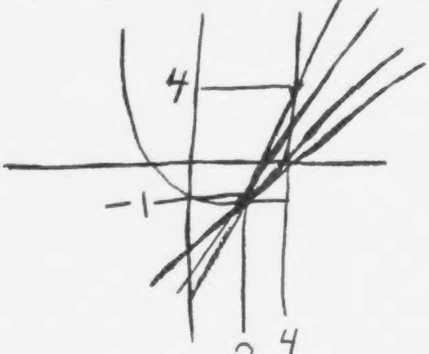
J: Ok. f x whatever, x plus, I don't know, x cubed plus five or whatever. f prime would be just 2 x plus x squared. Now that would just be the slope of a graph, and you just plot the graph or whatever. I've always seen derivatives as a way of expressing an algebraic expression in a graphical sense.

I: Could you do that? The graphical part?

J: Probably not.

The first two of the above excerpts reveal a feature common to interview responses at all three institutions. This feature is that students were often able to give valid explanations for problems when the problems were visually oriented by graphs. The problems in which this generally occurred were Problems 5, 6, 9, 11 and 12. In particular, students used a range of *technical* and *everyday language* related to the visual appearance of a graph, or the concepts of slope and tangent line. This language was used to describe features of graphs, serving as a *source of conviction* to construct explanations. The *technical language* frequently used included: "constant", "not smooth", "not increasing or decreasing", "slope of the graph", "slope of a function", "negative slope", "tangent line", "slope of the tangent", "slope of the tangential line", "tangent horizontal", "no tangent line", "curvature", "always decreasing" and "derivative is the slope". *Everyday language* used included: "doesn't go smoothly", "steepest", "gets higher", "kind of straight", "turns sharply", "plateau", "flat", "changes direction", "levels off", "tangents going in different directions", "evens out", "coming down", "slope largest", "like top of a curve", "not a straight line anywhere", "slope most severe", "slope up and to the right", "it stops", "gets higher" and "rise over run". Examples of students' use of some of these *technical* and *everyday language* phrases are given below. In all these examples the students construct their responses from visually oriented language and this *language use* serves as a *source of conviction* by which to construct problem responses.

> (Cindy)
> (Problem 6)
> [6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]



der= slope
slope = $\frac{\Delta y}{\Delta x}$

$4 \to 2 = \Delta x$
$4 \to -1 = \Delta y$

$\frac{\Delta y}{\Delta x}$ = slope of tangent

Derivative is the slope of the tangent line of a point x on a graph f(x).

**Figure 55. Cindy's Written Response to Problem 6**

> Ok. Anyways, if you want the derivative at this point, what you're going to be doing is taking the slope of the tangent line. And the tangent line is the line that only touches the graph at exactly this point. And so you're taking the slope of this line. Um. Derivative, ok, derivative equals slope. And slope is the change in y value over the change in the x value.

(Leanne)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]

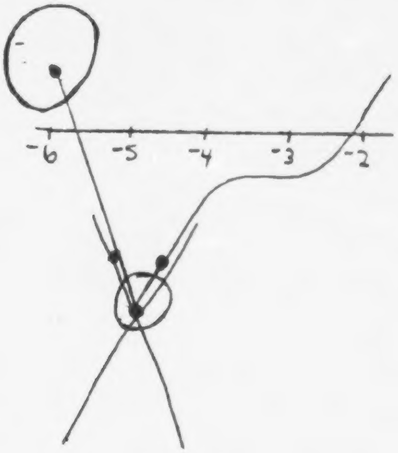


**Figure 56. Leanne's Written Response to Problem 9**

If you had a tangent on this side it would be going this way [draws a line]. And as it approaches the point from this side it would be going like this [draws another line]. And this would differ. Like you couldn't find a slope.

(Sally)
(Problem 11a)
[(a) At what exact point in time was the number of elk increasing most rapidly?]
(See page 65 for the graph for Problem 11)

The derivative is the slope of the graph, and so um, it says right here that the graph shows the number of elks in the park. So um, when the number is increasing over the shortest amount of time, then the slope is going to be the most vertical. And when it's the most vertical it's going to be the highest. And there it's the most vertical so it will be the highest there.

(Richard)
(Problem 11e)
[(e) At what point or points in time is the number of wolves not changing?]
(See page 65 for the graph for Problem 11)

I: Why where the derivative is zero?

R: Slope is, well the slope of the tangent line is zero. It's horizontal. So slope is zero. Well nothing is increasing or decreasing. There's no more wolves and no less wolves at either point. It just stops.

> (Jennifer)
> (Problem 11e)
> [(e) At what point or points in time is the number of wolves not changing?]
> (See page 65 for the graph for Problem 11)
>
> Well the line isn't increasing or decreasing. It's just maintaining a steady slope. It's flat.

In all these excerpts the students give explanations that rely on visual, spatially oriented language, including: "touches the graph at exactly this point", "going this way", "from this side", "most vertical", "highest", "horizontal", "just stops", and "flat". Through use of these *everyday language* phrases to describe the graphs under consideration the students integrate *everyday language* with the *technical language* terms tangent, slope, and derivative. Their explanations are thereby constructed from *everyday* and *technical language* as *sources of conviction.*

Before continuing with analysis of students' *language use*, consideration of whether *language use* as a *source of conviction* is external or internal in nature is in order. First, students' use of pre-calculus mathematical language and *everyday language* as *sources of conviction* has been demonstrated. It would be logical to consider this form of *language use* as an internal *source of conviction* because it seems logical that *everyday language* knowledge is personally meaningful to students. Second, use of visually oriented language as a *source of conviction* has been demonstrated. It would be logical to consider this language use an an internal *source of conviction* because students use it in conjunction with perceptual experiences to construct problem responses. However, there is not enough evidence at this point to claim that *everyday language use* within a mathematical environment is perceived by students to have personal meaning. Whether or not it has personal meaning would depend on the extent to which they perceive related mathematics conceptualizations to be personally understood. This issue of whether or not *everyday* and visually oriented language use as *sources of conviction* are external or internal in nature will be examined again in Chapter 5.

In ways similar to students at Alpha University and Beta College, students at Gamma College gave visually oriented descriptions and explanations. However, in addition to similar visually oriented language, they constructed a number of descriptions from the visually oriented notion of magnifying a curve. At some point in the interviews all students at Gamma College spoke of infinitely "magnifying" or "blowing up" the graph of a function. In an infinitesimal approach to calculus this is a means by which a function can be examined "up close". The following interview extracts exemplify how this process works. They also display related *technical* and *everyday language* and how it plays a role in the students' constructions of problem responses.

> (Neil)
> (Problem 6)
> [6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?]

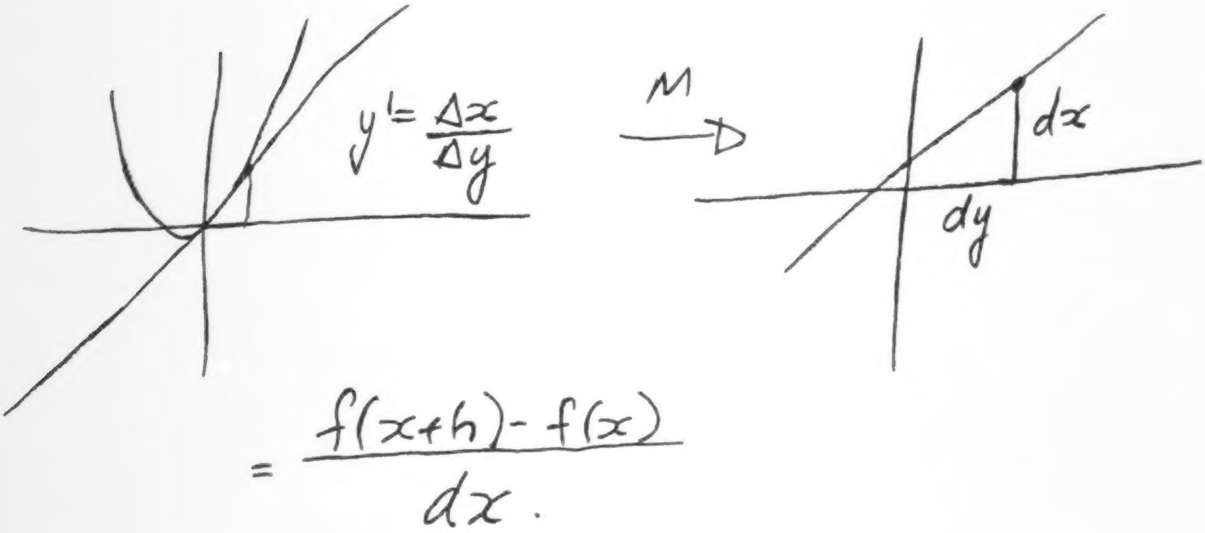


**Figure 57. Neil's Written Response to Problem 6**

> If you were to magnify that function infinitely it would look like a straight line with the same point. And you could still have a rise and a run. Except the rise and the run would be infinitesimal as compared to a finite rise and run.

(Gordon)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]

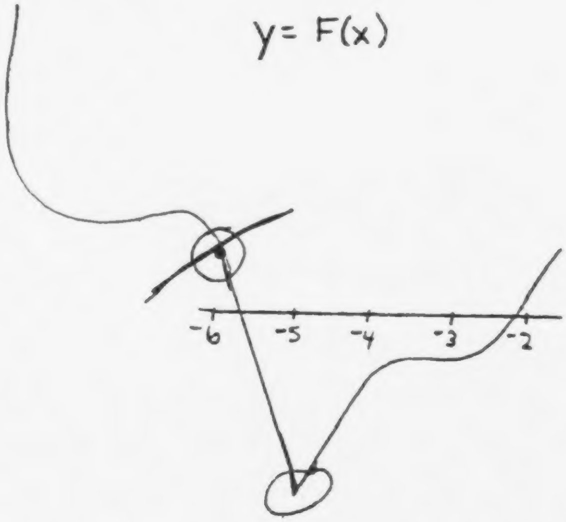


**Figure 58. Gordon's Written Response to Problem 9**

The line has to be continuous. So you wouldn't have one [a tangent] at the endpoint. [pause] If you blow that up, infinitesimally you still have that. You can't draw a tangent to that. Then you can't have a derivative.

(Betty)
(Problem 11e)
[(e) At what point or points in time is the number of wolves not changing?]
(See page 65 for the graph for Problem 11)

B: Just like here it's flat. But that's going from, when you would ah like increase it. Like increase the graph and focus on that it would be like more of a flat line as you get closer. And then flatter and flatter.

I: What do you mean by increase and focus on it?

B: When you magnify it.

(Tanya)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]
(See page 64 for the graph for Problem 9)

Right at this point if you magnify it. You're magnifying the point and you still have a straight line. In order to have a derivative you need a line. You don't need a point and a line to the left or right of it. You need a line where you can draw a tangent line and a slope to it. Here, like I said, a derivative just to the right of it exists [at x = 1/2]. Left, sorry. Just to the left it exists. Infinitesimally. Right at that point it doesn't exist.

(Nadine)
(Problem 9)
[9. The graph of y = F(x) is given below. At which points does the function not have a derivative? Why?]
(See page 64 for the graph for Problem 9)

. . . you take the point and you blow it up an infinite amount. And if you see a straight line there's a derivative. . . . You'll still see this. You blow it up and you'll still see a V [at x = -5]. And at this point there is no derivative.

(Mike)
(Problem 5)
[5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]

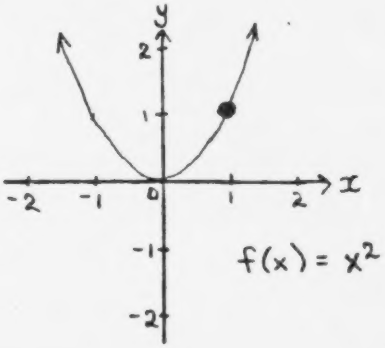


**Figure 59. Mike's Written Response to Problem 5**

When you're looking that close and like on any point whatever it would be easy to blow it up then and then see your dx. Like a little point almost on top of it. You can see it. It's going to blow up almost like a straight line. And it's going to be almost, it's going to be, it will round off. Like just to the right here. Just to the right. Like I said, infinitesimally close.

In the above extracts the students use both *technical* and *everyday language*. *Technical language* terms such as "infinitely", "infinitesimally", "continuous", "endpoint", "tangent", "derivative", "slope", and "straight line" are generally integrated with use of *everyday language*. This *everyday language* includes: "a rise and a run", "more like a flat line as you get closer", "a point and a line to the left or right of it", "see a V" and "blow it up". The language plays a role in students' interpretations by orienting the students to construct descriptions of a magnified curve. Thus, it can be said that infinitesimal language use related to magnifying a curve serves as a *source of conviction* by which students can construct graphical interpretations and justifications for these interpretations. Whether this *source of conviction* is external or internal in nature is not however clear at this point.

Non-infinitesimal language related to the slope of a tangent line also served to orient students to descriptions of a curve. However, these descriptions, justifications and conclusions seldom made use of limit-related language or processes. In comparison, the notion of infinite magnification has limiting processes built into its use. This feature distinguishes it from traditional slope and tangent line notions in more than one way. First, it is a dynamic rather than static method for interpretation of graphs. Second, magnification makes the limit concept of "close to" accessible. That is, the visual mechanism of blowing up or infinitely magnifying a curve serves as a visual, physically accessible means or *source of conviction* by which to examine limiting notions. The traditional limit concept also has visual interpretations, but these were not regularly used by students at Alpha University or Beta College. In fact, the general absence of use of limit notation or terminology by students at these two institutions, unless it was specifically requested, indicates the students did not integrate their limit conceptualizations into other calculus conceptualizations. For example, their responses to Problem 6 frequently included explanation of the derivative as the limit of the slopes of a sequence of secant lines, but the relationship of limits and derivatives was then not applied in other problem responses. Use of the notion of magnification was more regularly applied by students as a *source of conviction* by which to construct calculus conceptualizations.

A final feature of the Gamma College students' use of magnification was that when students used related terminology they did not construct the same misconceptions present in problem responses of students who did not use infinitesimal terminology (including incidents from Gamma College interviews when the students did not use infinitesimal terminology). For example, there were students at all three institutions who interpreted continuity in terms of the word "continuing". In conjunction with the *technical language* term "continuous" students used *everyday language* phrases such as: "no breaks", "no jumps", "existing", "being defined" and "not changing". Many of the students' notions associated with these *everyday language* phrases were valid interpretations of situations, although they were not necessarily valid mathematical interpretations. The interpretations therefore sometimes guided students to construct mathematically incorrect justifications, or justifications that were used inconsistently. For example, students' problem responses at all three institutions included:

> (Doug)
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
> (See page 61 for the graph for Problem 5)
>
> Well, cause there's no breaks [first graph]. It just continues. And all the points are, all the points exist on it. And they keep going. . . . I guess cause it changes direction [third graph]. It goes this way and then it just breaks. It doesn't have a, it doesn't smoothly go into it. It's like two different lines that just happen to start at that point. . . . No, it's discontinuous [fourth graph]. It exists but it's discontinuous.
>
> (Jennifer)
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
> (See page 61 for the graph for Problem 5)
>
> I just imagine you can just, using values in there [first graph], there'd be no values of x where the function wouldn't exist. . . . I'd just say any, it would be continuous because any number that x, the function would exist at any number x.
>
> (Gordon)
> (Problem 5)
> [5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
> (See page 61 for the graph for Problem 5)
>
> G: This one is [first graph]. Cause the arrows indicate that it's going on forever. There's no ah, no empty spaces between it. Then you could say it is continuous. Um. [pause] This one is too [fourth graph]. Because everything to the left of that point is on the line. And everything on this line to the right is there. There's no empty spaces. Except for that little open circle which is there. If you just looked at it, as like a straight line, it's still going to be, everything is still going to be there. There's not going to be, no empty spaces.
>
> I: That's in this part here? [the discontinuity at $x = 1$]
>
> G: Cause like if you brought this line [the right half of the fourth graph] down there, the open circle and the closed circle would just cancel each other out. And you'd just have a straight line.

I: So are you saying it's continuous?

G: Yeah, it is.

In these three extracts the students' interpretations of the *technical language* term "continuous" in terms of "existing" leads the students to construct mathematically incorrect justifications. Further, although interpretation of "continuous" as "no breaks" usually oriented students to mathematically correct notions related to continuity, it did not do so for Doug. Doug believed that a "break" in the way a function is defined constitutes a discontinuity.

Another misconception displayed by students who did not use the notion of magnification was that non-uniqueness rather than non-existence of a tangent line implies non-differentiability. For example, students said such things as the following:

(Cindy)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?] (See page 64 for the graph for Problem 9)

Because this is undefined [at $x = -5$]. Because a derivative means you're taking the slope of a tangent. But the tangent, it could be here, it could be here, it could be here. It could be anywhere. And we don't know where it is.

(Annabel)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?] (See page 64 for the graph for Problem 9)

Derivative is suppose to ah, on a graph the derivative is supposed to be a tangent line that touches the graph at only one spot. And at a sharp point or an endpoint there it touches it, it can do that in many different places. So you cannot define any one derivative.

(Richard)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?] (See page 64 for the graph for Problem 9)

Well I mean a cusp, this is the way a cusp is drawn [at $x = -5$]. But if you get a cusp it could go like that. And you don't know what, can you draw like that? How do you draw, you can draw it a lot of different ways.

(Sally)
(Problem 9)
[9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?] (See page 64 for the graph for Problem 9)

Like the tangent to this line. There's not really a place where you could put a tangent [at $x = -5$]. It could be here, or here. So there's no tangent for the derivative to be equal to.

The students to whom the above interview excerpts belong were guided by their responses to the correct conclusion that no derivative existed at a particular point. In comparison, magnification of a curve generally served to focus students' perceptions and subsequent justifications upon non-existence rather than non-uniqueness of a tangent line. Examples of this have already been given (see Gordon, Tanya and Nadine's interview excerpts on pages 226 and 227).

Amongst the problems that have not featured prominently in these discussions thus far, Problems 3a, 4, 7, 10 and 12, Problems 3a, 4 and 7 were most useful in revealing students' *language use*. Problems 10 and 12 were not as useful in revealing *language use* because students did not as extensively vocalize their written responses to these problems. In addition, due to time constraints in conducting the interviews, responses to these problems were not generally probed as extensively as other problem responses.

Since solutions to Problems 3a, 4 and 7 require symbolic operations, these problems provided opportunity for examination of students' *language use* while performing symbolic procedures. To begin with, 14 of the 17 interview students spoke of the limit or round off situation in Problem 3a in terms of the relative size of numbers. They said such things as:

(Doug)
(Problem 3a)
[3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5} ]$$

It's just a really really big number over a really really big number that's not so big.

(Cindy)
(Problem 3a)
C: Ok. Well, these two are, they're not really worthy of being noted because four and five aren't really going to make much of a difference tacked onto an infinity. So if you put infinity in here [where there is an x]. Let's see. [pause]

I: What are you thinking?

C: I'm thinking that, my immediate reaction is to go like this. And that makes a large number over a large number. But this number up here is going to be larger than this number is. And so you're going to end up with a large number. Like it's going to equal infinity.

(Annabel)
(Problem 3a)
I guess you know that it's still going to equal infinity because infinity to the fourth is obviously larger than infinity to the third minus infinity.

(Gordon)
(Problem 3a)
[3. (a) Round off the following:

$$\frac{M^4 + 4}{M^3 - M + 5}]$$

Because these ones are so much higher than that one you can just forget them. [pause] So you just get M. [pause] And finite divided by an infinitesimal, or I mean divided by an infinite is just infinitesimal. . . . This one here is a finite number. You can just forget about that again. A finite number divided by that is so small you can just forget about it. So it rounds off to M.

(Tanya)
(Problem 3a)
Because this would be an indeterminate amount. You can't really see what's happening because this is infinity over infinity. And that really doesn't say anything. . . . Once again dx stands for an infinitesimal. Any finite over an infinite is an infinitesimal.

(Neil)
(Problem 3a)
That term is an infinitesimal, and so are these two. So it's infinity over one.

In all these responses the students base their arguments on the numerical magnitude of numbers. What is distinctive is that the Gamma College students' *language use* is more *technical* in nature, using the term "infinitesimal" to denote a very small number. Further, three of the six interview students at Gamma College referred to the symbolic expression in Problem 3a as an indeterminate form. In comparison, none of the other 11 students spoke of the corresponding limit notation expression as an indeterminate form. This fact is especially noteworthy in conjunction with the fact that all six students at Gamma College used the term "indeterminate" to describe the function in Problem 4 at the point $x = 2$. Only one of the other 11 interview students made a similar statement (Ned), saying it was "undetermined". Seven of these eleven students substituted $x = 2$ into the denominator only, or sometimes the whole expression for the function. They then concluded that because "you can't divide by zero" the function is undefined at $x = 2$.

The reasons for this difference amongst students as to their recognition of an indeterminate form are not clear. It might be that instruction using rounding off interpretations guides students to focus on the form of an expression, rather than immediate focus upon performance of symbolic operations. When students at Gamma College are taught the rounding off process the notion of the relative size of a symbolic expression is emphasized. First, symbolic expressions are examined as to whether their component parts are infinite, finite or infinitesimal in size. The overall size of the expression is then determined. Although algebraic operations might be needed before a final decision is made, the emphasis is still upon size. The limit concept has similar interpretations, yet it appears that students at Alpha University and Beta College conceptualized limits in ways that are not as applicable as are Gamma College students' conceptualizations of rounding off.

Students' *language use* in Problem 7 was almost strictly in terms of verbalizing the steps they were going through to determine the derivative. Examples of this *language use* are:

> (Doug)
> (Problem 7a)
> [7. Find the derivative of each of the following:
>
> $$y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}]$$
>
> You take the derivative of the numerator times the denominator, and you minus the numerator times the derivative of the denominator. And you put it, but you have to get the derivatives of these.

(Sally)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7 ]$

So first you have the inner function. And the outer function. So you get 10 [mumbling] times 4 t plus 3. And then 3t. And then you use the multiplication rule for derivatives. Which is the derivative of the first one times the second term. Plus the first term times the derivative of the second term. So then we took the derivative of the first term. And then there's the second term. And then plus, and then this normal term. Times the derivative of the first term. So we first take the outer function. Times, and then the inner function.

(Tanya)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7 ]$

This is the product rule. We're doing the derivative of this times this. Plus the derivative of this times this. Derivative of this is ah derivative of outside the function. Which brings the ten down and like a power rule with the brackets. Times. This is the chain rule. Times the derivative of what's inside. So that takes care of the derivative of this. And times this part. Plus, as I said, the derivative of this is once again doing the product rule with brackets. Times the chain rule in here.

(Richard)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7 ]$

I used the product rule. The first, I did the derivative of the first times the second. So the derivative of the entire function. And then multiplied by the derivative of what's inside the function.

(Ellen)
(Problem 7b)
$[F(t) = (2t^2 + 3t - 2)^{10} (3t^{1/4} - 9)^7 ]$

But then you have to find the derivative of the first using the chain rule. . . . Took the derivative of the second one and then found what the derivative of the inside was again. And then timesed it by just this normal equation.

In the above excerpts, whether using *technical language* such as "derivative of the numerator" or *everyday language* such as "outer" or "inner", students' *language use* in determining a derivative served as a recipe. Following the directions of the recipe achieved an answer. In other words, students' *everyday* or *technical language use* that verbalized symbolic procedures served the students as *sources of conviction* by which to construct problem responses. The same procedural use of language was evident in responses to

Problem 10 for those students who verbalized their implicit differentiation steps (for example, see Richard and Mike's Problem 10 responses on pages 205 and 206). As with *everyday language use* it is not presently clear if this procedural use of language is external or internal in nature. More specifically, although it is clear students' procedural use of language is a *source of conviction* by which to construct calculus responses, whether or not students perceive this use to be personally understood is not clear.

In summary, students at all three institutions were similar in the extent to which they used *technical language*, but Gamma College students used more *everyday language*. In addition, the content of Gamma College students' *technical* and *everyday language use* was different from the other students'. They used *technical* and *everyday language* related to infinitesimals, as well as related visual notions such as the location of two infinitesimally close points and infinite magnification. Their language use thereby served them as a *source of conviction* from which to build calculus conceptualizations. Students from Alpha University and Beta College did not generally display as much integration of *technical* and *everyday language* related to calculus. All students did however display use of pre-calculus language knowledge as a *source of conviction*. Gamma College students carried previous knowledge of the term "round off" into the formation of their calculus conceptualizations. Alpha University and Beta College students used knowledge of the term "limit".

Many of the students displayed misconceptions of calculus concepts when they did not appropriately integrate *everyday* and *technical language*. However, when students used infinitesimal language they did not display the misconceptions present in other students' responses. They more frequently appropriately used *technical* and *everyday language* in conjunction with one another. Finally, all students were often able to give valid explanations for visually oriented notions and notions related to relative size. They used these notions and related language as *sources of conviction* from which to construct problem responses. In addition, for students from all three institutions, language knowledge arising from symbolic procedures served as a *source of conviction*. However, whether it be previous or *everyday language use*, visually oriented language use, or procedural language use, it is not clear if language use as a *source of conviction* is external or internal in nature. More research is needed before this decision can be more definitively determined, an issue that will be discussed in Chapter 5.

### Figures and Objects

The values in Table 10 under the figures (F) and objects (O) columns indicate that students at all three institutions made little use of physical objects in their calculus responses. They also indicate that, on average, students at Beta College used more figures in their problem responses. This fact means that students at Beta College introduced their own graphs or diagrams into problem responses. The actual numbers indicate that on average they incorporated 2 or 3 more figures into responses than did the other students. However, more prominent distinctions from the other students are evident in examination of when they introduced figures and what they said in relation to these figures.

To begin with all Beta College students except Doug introduced figures in at least one of their explanations for Problems 2 and 3b, problems involving limits. These were the problems that explicitly asked students to explain or justify limits. In comparison, none of the other 11 students used figures in their explanations for Problem 2, and only 3 of the 11 used figures in their interpretations for limits in Problem 3b. Some examples of Beta College students' use of figures in Problems 2 and 3b are:

> (Tim)
> (Problem 3b)
> [(b) What does "limit" mean to you?]

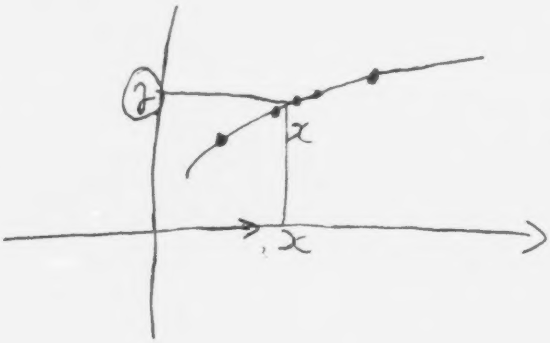
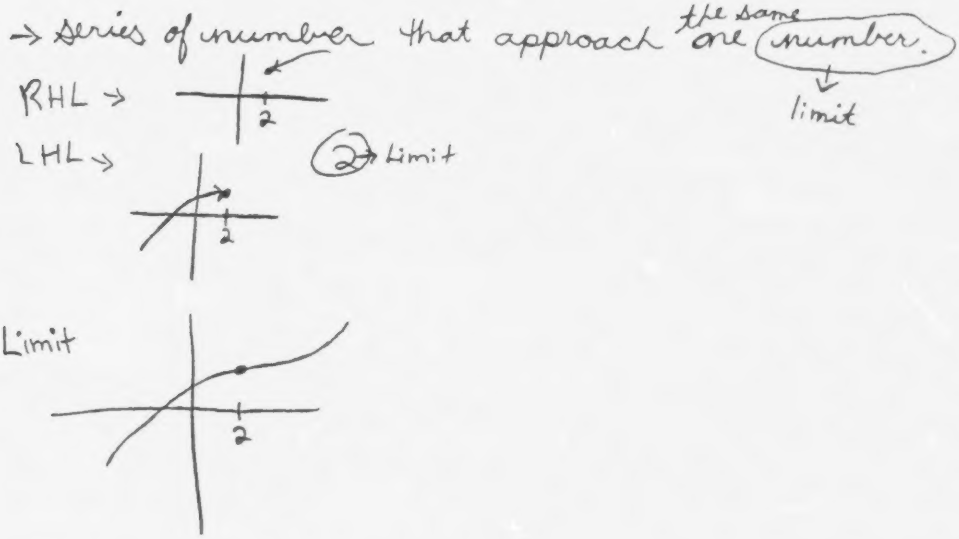


**Figure 60. Tim's Written Response to Problem 3b**

> I think I will say, um I will explain this by drawing. If there is a value of x, um, we define another point just close, close to it. Approach the value. Approach the point x. And then we try to, ah, we try to put, we try to shift the point which approach to x closer and closer. Closer and closer, and we get the value, ah. This is x there. This is y. We try to find this value which closer and closer to get the value, ah, to see whether this move to ah x to find the answer. A certain number.

(Cindy)
(Problem 3b)
[(b) What does "limit" mean to you?]



→ series of number that approach the same one number.

Figure 61. Cindy's Written Response to Problem 3b

Okay. For a right-hand limit is ah is, I'm going to draw a picture for this. The limit, okay, say this was two and this was one. That's the graph approaching two from the right-hand side here. And the left-hand limit is two again. The graph approaching two from the left-hand side.

(Leanne)
(Problem 2)
[2. For each of the following sequences of numbers, decide whether the sequence has a limit. If so, what is this number?

$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$

3.9, 3.99, 3.999, 3.9999, 3.99999, 3.999999, . . .]

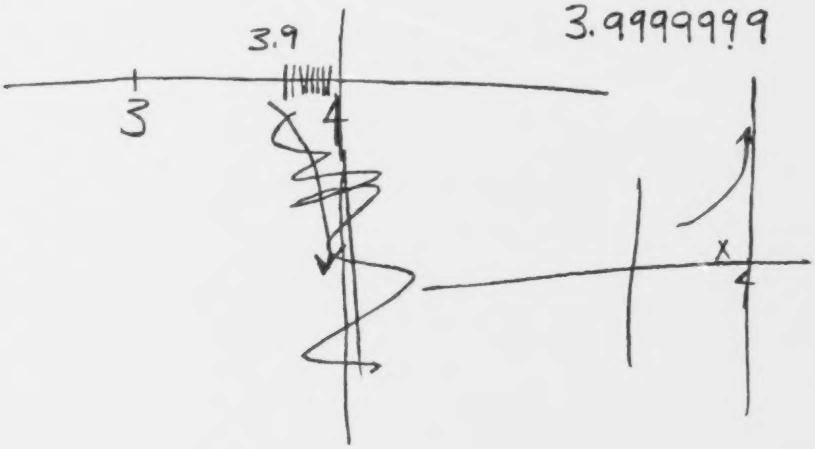


**Figure 62. Leanne's Written Response to Problem 2**

If you had a line graph maybe. And say this was three. But ah, you'd be continually getting closer but never reach four I guess. I guess you could draw it like that. . . . Like if a curve were approaching a number four, that will never, like if, if you had like a graph it could be either a horizontal or vertical curve. And um it will always be approaching the number four. And it's x value would be three point nine, but it will never actually reach there.

(Daniel)
(Problem 3b)
[(b) What does "limit" mean to you?]

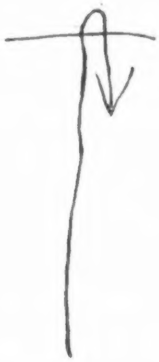


Figure 63. Daniel's Written Response to Problem 3b

> I said here it represents a barrier or endpoint. But ah, I guess if you're thinking in terms of something rising, yeah, it could go beyond that and come back.

In all these excerpts students use figures to interpret limits. This fact, along with the fact that Beta College students tended to give lengthier descriptions and explanations of figures than most of the other students indicates Beta College students language use related to figures was different in content to most of the other students' figure descriptions. Examples of their descriptions of figures included:

(Daniel)
(Problem 5)
[5.For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
(See page 61 for the graph for Problem 5)

> Well there's no break in the graph and it appears to go to infinity like this forever [first graph]. And that is further demonstrated by the function x squared because, what do you call it, exponential growth? Like it will cause this thing to go on forever. And also from looking at the graph I see its smooth. There's no breaks or spaces where the graph doesn't exist. . . . I find this one tricky [second graph]. The one with the two hyperbolas. Because ah, um, on one hand I think the graph is smooth. Both the hyperbolas are smooth and approaching infinity. But there is ah, but x can't equal zero. Which seems to me to provide a discontinuity. But I don't think that provides a very strong argument so I'll say that is continuous. . . . I remember reading in my calculus textbook something about continuous functions are smooth with no breaks. And I was trying to remember if it also included, what do you call it, ah, sharp turns in a graph. Because I believe that makes it discontinuous at this part right here [third graph]. Because it's stopping and taking another direction. . . . To me what really is a blatant discontinuity is a hole, a cusp.

> Which I believe a cusp is. . . . It seems to me, I don't see why this wouldn't be continuous [third graph] because it's going off in one, like it's just taking one direction and going off in another. It's not a hole in the graph. But I believe that when I look at it, maybe the whole fact that it is taking another completely new direction shows the end of one function, or the end of one set of requirements and the beginning of another. Without flowing together. Like there's a disjoint. For this one I would call it discontinuous because there's a hole in the graph at one here [fourth graph]. And here, ah, at negative one, this is both on the x-axis. It exists at one. Like that's where it starts. Like it doesn't look continuous. I suppose down here [for x<1] it does become continuous as ah it starts and goes forever. There's no place on this it doesn't exist. And here it does not exist at the coordinates one comma two. There's a hole in the graph and therefore it's a discontinuous function.
>
> (Sally)
> (Problem 5)
> [5.For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.]
> (See page 61 for the graph for Problem 5)
>
> This one is continuous because it never stops [first graph]. There's no gaps. And there's no, like you could just keep drawing it forever. . . . This one's not continuous because you're here and then here you have to lift your pencil [second graph]. And then you can go again. . . . Then this one's continuous [third graph]. It's just not smooth. And it's continuous for the same reasons . Cause you could just draw it and draw it forever and ever. . . . Well, you can keep going and like there's no breaks. No breaks in the graph. No breaks on the function. The function keeps going. And this one's discontinuous again [fourth graph]. Because here's the graph down here. And you can draw it here and then you have to jump again to here. And then as soon as you reach here you have to jump again to here.
>
> (Leanne)
> (Problem 9)
> [9.The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?] (See page 64 for the graph for Problem 9)
>
> At negative 6 it doesn't have a derivative. . . . Because it's an endpoint. And the derivative can't be on a closed interval. Um. I can't think of why. And here because, um it's not smooth and continuous. If you were to draw a tangent to this part of it. . . . And if you had a derivative here it would be going a different direction. Whereas if it were smooth and continuous you could find two points to, um. . . . If you had tangent on this side it would be going this way. And as it approaches the point from this side it would be going like this. And this would differ. Like you couldn't find a slope. There's no real, well you could find a horizontal slope. I'm not sure exactly. Here, a straight line, the derivative is zero. . . . I don't think there's one here. It's a sharp point in the graph. And there isn't one at five. Cause the graph is undefined at five. And that's it.

These three interview extracts demonstrate the length of many of Beta College students' responses to problems that gave graphs in the problem statement. They also demonstrate the physical nature of these responses. This physical nature is seen in the students' descriptions of the graphs using such phrases as: "go on forever", "no breaks", "smooth", "sharp turns", "stopping", "taking one direction", "hole", "flowing together", "no gaps", "lift your pencil", "draw it forever and ever", "you have to jump", "different direction", "going this way" and "sharp point". Although this use of language related to bodily, physical experiences is demonstrated here with Beta College students' problem responses, it was also evident in other students' problem responses. For example, see Doug, Jennifer and Gordon's problem responses on page 229.

It is therefore seen that visual or physical interpretations of calculus are means by which students can build problem responses and related conceptualizations. Since these interpretations are a foundation upon which to construct descriptions and explanations they serve students as *sources of conviction*. Whether students perceive of these physically oriented descriptions as personally meaningful within a mathematical context is not clear. It is therefore not clear if use of figures and related language as *sources of conviction* are external or internal in nature. Constructivism is evident is these descriptions and explanations in that through use of *everyday* and *technical language* students construct individual responses. However, the extent to which students have personal understanding of their related conceptualizations is not evident. It would seem logical to conclude physical interpretations are internal in nature, because physical experiences are inherently personal, but there is insufficient evidence at this point to support this claim.

# 5. SUMMARY, DISCUSSION, IMPLICATIONS AND RECOMMENDATIONS

This study was designed to investigate student learning in calculus from a constructivist perspective. The nature of students' constructivist learning was examined through students' *language use* and *sources of conviction.* Data collection and analysis were guided by the following areas of inquiry: (1) the nature and role of the language students use to interpret calculus concepts and problems, (2) the nature and role of students' convictions regarding the validity or truth of calculus interpretations and problem responses, (3) the ways students construct their calculus conceptualizations, (4) the ways different approaches to calculus instruction appear to impact on students' *language use, sources of conviction* and manner of construction of conceptualizations, and (5) the ways technique-oriented, concepts-first and infinitesimal approaches to calculus instruction translate into classroom, and textbook and exercise assignment instructional events.

## A. Summary

This research was a naturalistic study involving three undergraduate calculus classes, each being taught from one of three instructional approaches: technique-oriented, concepts-first and infinitesimal instruction. Clinical and personal interviews at the end of the term with 17 students were the primary method of inquiry into students' *language use, sources of conviction*, and manner of construction of conceptualizations (research areas (1), (2), (3) and (4)). Instructor interviews, classroom observations done over a 13 week school term, and instructional materials analysis provided description of each instructional setting in terms of *language use* and *sources of conviction*. These activities allowed research areas (4) and (5) to be addressed. The following points summarize the findings of the student interview analyses. These points summarize the findings of the first three areas of inquiry, students *language use*, *sources of conviction* and manner of construction of conceptualizations. The fourth and fifth areas are summarized in Section D of this chapter, Implications for Instruction.

### The Nature and Role of Students' *Language Use*

Symbols do not form a large component of student's *language use* in calculus. Although students are often able to manipulate or perform operations with symbols, they do not generally use symbols to interpret concepts. However, Connector students displayed more extensive use of symbols, using them to explain or justify calculus ideas and problem responses.

Students from all three institutions were similar in the extent to which they used *technical language*, but the content of Gamma College students' *technical language* differed from the other students, and it was integrated more with *everyday language*. Gamma College students used infinitesimal language and related visual notions such as infinitesimal closeness and infinite magnification.

All students displayed use of pre-calculus mathematics language knowledge as a *source of conviction*, and many students displayed misconceptions when they did not appropriately integrate *everyday* and *technical language* (including symbols). Students who used infinitesimal language, more frequently than the other students, appropriately integrated *technical* and *everyday language*. All students often gave valid physical interpretations for visually oriented notions such as continuity, slope or size. They used visually oriented language as *sources of conviction*. In addition, many students displayed procedural use of language as a technology or *source of conviction* by which to construct problem responses.

### The Nature and Role of Students' *Sources of Conviction*

From analysis of students' *sources of conviction* as revealed in their problem responses and comments on their own learning three main groups of students emerged. These three groups, Collectors, Technicians, and Connectors, differ from each other in the degree to which their *sources of conviction* are external or internal in nature. The role of their *sources of conviction* in the construction of calculus conceptualizations also differs.

Collectors exhibit the highest degree of external *sources of conviction*, with their *sources of conviction* originating from teacher or textbook presentations of statements, rules and procedures. The role of a Collector's *sources of conviction* is as a validation to the student that she or he makes statements and performs procedures that individuals believed to be knowledgeable in mathematics will see as valid or correct.

Technicians display a mixture of external and internal *sources of conviction*. Their external *sources of conviction* are based on knowledge of calculus statements, rules and procedures, while their internal *sources of conviction* reside in a personal sense of mastery of calculus as a technology by which to think about and apply concepts and procedures. The role of their *sources of conviction* is as a set of tools that can be employed to think about and apply calculus statements, rules and procedures.

Connectors display the highest degree of internal *sources of conviction*. They exhibit knowledge of statements, rules and procedures, and the technology of calculus as *sources of conviction*, but they also display a sense of personal understanding or ownership of their calculus conceptualizations. The role of a Connector's *sources of* *conviction* is as a validation to the student that he or she makes statements or performs procedures that are meaningful to himself or herself as well as other individuals.

In relation to *sources of conviction*, additional major findings of this study are: (1) valid examination and determination of students' *sources of conviction* must include students' perceptions of their learning and (2) *language use* is a source of conviction.

### The Ways Students Construct Calculus Conceptualizations

Collectors use language and external *sources of conviction* to construct their calculus conceptualizations as a collection or assemblage of isolated, relatively unconnected mathematical statements, rules and procedures. Although they use language knowledge as a *source of conviction*, particularly in relation to visually oriented mathematical representations, they do not necessarily acknowledge the place of these personal interpretations in the construction of their conceptualizations.

Technicians use language knowledge and a mixture of external and internal *sources of conviction* to construct their calculus conceptualizations as a logical organization of statements, rules and procedures. That is, their calculus conceptualizations are built as a technology or method for thinking about and applying calculus. Through knowledge of how to solve problems by the use of calculus symbols and terminology Technicians structure and organize their calculus conceptualizations. These students thereby become skilled users of calculus language.

Connectors construct their calculus conceptualizations as a network of connections between various aspects of calculus and between calculus and themselves. They display some *sources of conviction* similar in nature to Collectors' or Technicians' in that they arise from teacher or textbook presentations or knowledge of calculus as a technology. However, they integrate these *sources of conviction* and related conceptualizations into conceptualizations they develop through their own thought processes and interactions with calculus material. Thus, Connectors' conceptualizations are built as entities of which they display a sense of personal understanding.

## B. Reflections on the Study's Evolution

This study began with an aim to study student learning in introductory calculus from a constructivist perspective. The adoption of a constructivist perspective pointed to *language use* and *sources of conviction* as means by which to examine student learning, while simultaneously testing and refining constructivist notions. *Language use* was seen as important to learning viewed from a constructivist perspective because constructivism sees mathematics knowledge as being grounded in subjective interpretations of language (von

Glasersfeld, 1987a; p.7) and "linguistic knowledge, conventions and rules" (Ernest, 1991; p.42). *Sources of conviction* were deemed as important to learning viewed from a constructivist perspective in that constructivism sees knowledge objectivity as residing in a social sharing of decisions pertaining to truth and validity. Thus, *language use* and *sources of conviction* were adopted as a focus for this study. Since this study was undertaken in three different instructional settings it was able in a range of environments to examine and develop constructivist notions through the related notions of *language use* and *sources of conviction*.

The complex nature of students' *sources of conviction* in terms of the necessity of integrating personal interview responses into analysis of clinical problem responses was not evident at the outset of this study. The inter-connectedness of *language use* and *sources of conviction* was also not apparent at that time. These factors became apparent only once students' interview transcripts were extensively and critically examined. Pilot study work had indicated students used a variety of *technical* and *everyday language*. It had also demonstrated the range and interspersion of students' *sources of conviction*, including reference to the teacher, the textbook, the physical structure of the world, the structure and rules of mathematics, or a student's personal perceptions and beliefs. In the initial development of *language use* and *sources of conviction* (see Chapters 2 and 3) it was conceived to be both practical and valid to classify students' individual statements as indicative of a particular language context, or external or internal *source of conviction*. Although these classifications were applied to description and analysis of classroom instruction and textbook and exercise materials, and proved to be useful and informative within these contexts, they encountered difficulties when applied to analysis of student interviews. The overall picture of a student's *sources of conviction* as revealed by the student's comments on her or his calculus learning showed the *source of conviction* used by a student in a specific statement was not independent of the student's views of her or his learning. In particular, the classification of mathematical or experiential (physical) knowledge as necessarily internal in nature conflicted with students' views of their own learning. For example, many students justified their problem responses through reference to their mathematics or experiential knowledge, while concurrently declaring they memorized or did not understand particular statements, rules or procedures.

This last fact is an important finding of this research study. It is important in that it reveals the necessity and importance in mathematics education research of consideration of students' own perceptions of their learning. In addition, it was only through examination of students' comments on how they viewed calculus and how they approached their calculus learning that the three groups of students, Collectors, Technicians and Connectors emerged. The emergence of these three groups is another key finding of this study. The fact that ***language use*** serves students as a ***source of conviction*** was also apparent only after examination of what students said in problem responses, and what they said about their perceptions of their calculus learning. Thus, the fact that ***language use*** is a ***source of conviction*** is another key finding of this study.

The ultimate merging of ***language use*** and ***sources of conviction***, along with the emergence of the importance to learning of students' perceptions of their learning must be viewed as more than supplementary findings to the main research questions of this study. Both these findings add new dimensions to constructivist theory as applied to mathematics learning. They clarify and refine the notions of ***language use*** and ***sources of conviction*** as interpreted by constructivism. Further, since the study was conducted with students from three different instructional settings, they demonstrate the validity and usefulness of employing ***language use*** and ***sources of conviction*** as reflectors of learning viewed from a constructivist perspective. Hence, the validity and usefulness of constructivism for research into student learning in calculus is also demonstrated. End results of these developments are new methods for studying mathematics instruction and student learning in mathematics, and theory and related methods for incorporating constructivism into mathematics education research. Implications of this research study for constructivist theory and the use of constructivist theory in mathematics education research will now be discussed.

### C. Constructivism and the Research Literature Revisited

Over the past decade constructivism has emerged as an important influence in mathematics education research. Constructivism is a theory of knowledge that views mathematics learning as an active, constructive process in which an individual builds up knowledge for himself or herself. In particular, constructivism sees learning as an adaptive process in which an individual constructs a viable model of the world (von Glasersfeld, 1987a). Constructivism conceives of these models as a fit of knowledge to experience, rather than a match between knowledge and reality. In other words, constructivism views learning as "organizing experience so as to deal with a real world that cannot itself be known" (Kilpatrick, 1987; p.7). From this perspective, concepts are not viewed as entities which can be transferred "ready-made" from teachers to learners. Instead, constructivism sees learning as an individual process of making sense of new information by relating it to and reorganizing conceptual structures and processes.

This research study viewed student learning from a constructivist perspective. In adopting this perspective it was assumed that students construct their own, individual calculus conceptualizations. The central ideas of constructivism as outlined above proved to be a valid perspective by which to interpret students' calculus problem responses. That is, students' problem responses could be interpreted from a constructivist perspective in ways that make sense. They make sense in that they provide descriptions of students' responses that inform an observer of features that appear to be important components of the nature of students' calculus conceptualizations. For example, there were numerous events in each student interview that could be described as the student's individual construction of a viable model of his or her experiential world. For example, see Daniel's response to Problem 6 (page 149), Cindy's response to Problem 5 (page 150), or Jennifer's response to Problem 9 (page 140). There were also instances for which student responses could be interpreted as involving making sense of new information by connecting it to previous knowledge. For example, see Mike's response to Problem 6 (page 188), Gordon's response to Problem 3b (page 232), or Daniel's response to Problem 3b (page 239). In particular, students were seen to use pre-calculus and other *everyday language* knowledge as *sources of conviction* by which to make sense of information in their experiential worlds.

However, although the central ideas of constructivism could be used to interpret students' calculus responses, they did not immediately inform as to means by which to describe similarities and differences amongst students. The importance of identifying similarities and differences amongst students is that they can guide future instruction and research. It was only through the evolution of the constructivist related concepts of *language use* and *sources of conviction* that patterns emerged from student interview data. Evolution of the concept of *sources of conviction* led to emergence of three groups of students that differ from each other in the nature and role of their *sources of conviction*, while evolution of the concept of *language use* led to distinctions between students in the nature of their *technical* and *everyday language use*. In other words, the outcomes of this study, through development of methods for studying calculus instruction and learning, have led to further development of constructivist theory and means by which constructivism can be incorporated into mathematics education research.

This study also enhances constructivist theory in that it reveals an oversight in constructivist literature. Constructivist literature speaks of an individual's construction of viable models of her or his experiential world. Since it generally speaks of these constructions as personally meaningful to the constructor, it is incomplete in the ways it speaks of knowledge. It is incomplete in that individuals do not necessarily perceive their conceptualizations to be personally understood. More specifically, through the emergence of groups of students, in particular Collectors, this study reveals that students' conceptualizations are not necessarily the personally meaningful constructions of which constructivist literature in mathematics education often speaks. Of particular note in relation to this point is the fact that over half the interview students were classified as Collectors. These Collector students, although engaged in conceptual constructions, did not claim any personal understanding of their calculus conceptualizations. Thus, this study informs as to the nature of students' construction of their calculus conceptualizations. It thereby leads to a fuller understanding of student learning in calculus. This outcome in turn has implications for theory and related methods for facilitating calculus learning as a meaningful endeavour. These implications are discussed in the next section of this chapter.

An aspect of constructivism that must be addressed at this point is radical constructivism's rejection of realism. This study demonstrates that the central notion of constructivism, that individuals construct conceptualizations that are viable models of their experiential world, is a practical, appropriate means by which to describe students' calculus responses. However, this study's findings show radical constructivism's rejection of realism is not necessarily in accordance with the ways students view calculus and their calculus learning. Students do not generally see mathematical knowledge as lying in the "shared rules conventions, understanding, and meanings of the individual members of society, and in their interactions" (Ernest, 1991; p,82). For example, Collectors perceive calculus to be separate from their own reality and own understandings. From a Collector's perspective there is no negotiation of calculus knowledge through interaction with other individuals. Instead, calculus learning through a Collector's eyes is a matter of replication of externally generated and independently existing statements, rules and procedures.

Even for Technicians and Connectors it is not clear if radical constructivism appropriately describes the ways students perceive of their mathematics knowledge. Technicians' use of the technology of calculus as a *source of conviction*, along with the mixed nature of their external and internal *sources of conviction* indicates that to a certain degree they see calculus as a body of knowledge that exists independently of their interactions with it. Even for Connectors, since Connectors also display some external *sources of conviction*, it is not clear if radical constructivism adequately describes mathematical knowledge. Since Connectors display a sense of personal understanding of calculus conceptualizations, their calculus knowledge can be interpreted as subjective knowledge of objective or public knowledge (Ernest, 1991). In spite of this interpretation it must be recognized that most of the students were not Connectors. Therefore, the findings of this research study imply there are aspects of constructivism that are not appropriate for description of students' views of their calculus knowledge. These findings do not however imply radical constructivism has no practical value for mathematics education research. In fact, radical constructivism points to a means by which to develop theory and methods for facilitating calculus learning as a meaningful endeavour. These implications and others are discussed in the next section of this chapter.

A number of findings from this study also both support and develop previous theories related to mathematics learning. In particular, the role of language in mathematics learning is highlighted by this study's findings. First of all, Johnson's notions of "image schemata" are evidenced in this research study (Johnson, 1987). Image schemata are structures of meaning that arise from "perceptual interactions and bodily movements within our environment" (p.19). The image schemata that were evident in this study are those of continuity, slope, size and magnification. Students from all three institutions made use of physical experiences of continuity, slope, size or magnification. For example, in relation to continuity, students used the bodily experiences of existence, gaps, holes, jumps, and changes. In conjunction with the term slope they used the terms and visual notions related to increase, decrease, horizontal, vertical, steep, flat, and level. Size as an image schemata appeared when students referred to numbers or other entities. They gave descriptions involving such terms and notions as smaller, bigger, really really small, infinitesimally close, approaches infinity, or approaches a finite number. Finally, magnification and its related notions of shape and closeness were used as image schemata by which to describe and explain continuity and differentiability.

Another aspect of language and learning discussed in the literature that this study supports is the role of natural, everyday language (Halliday, 1978; Pimm, 1987). Students' previous language experiences influenced their calculus conceptualizations. For example, students' pre-calculus knowledge of the terms limit, round off, continuous, and undefined were evident in their calculus conceptualizations. This research study therefore further demonstrates that an individual's use and interpretation of *everyday language* is likely to figure in that individual's mathematics learning. In addition, students' use of the technology of calculus can be interpreted as use of the calculus "register" (Halliday, 1975; Pimm, 1987). It is use of a register in that it involves the *technical language* of calculus, as well as characteristic modes of arguing.

The role symbol systems play in learning is also evidenced in this study. Students' use of infinitesimal notation and related terminology was distinctly different from students' use of limit notation and related traditional terminology. Students who incorporated use of infinitesimal notation into their calculus problem responses used symbols as a "combined label and handle for identifying and manipulating concepts" (Skemp, 1987; p.62). They used infinitesimal language (words and symbols both) as tools by which to describe, explain and justify particular mathematical situations. In these instances their *language use* was a vital component of the construction of problem responses and related calculus conceptualizations. It was vital in that it oriented students to mathematically valid and useful descriptions, explanations and procedures.

West and Pines' notions of intuitive and school knowledge were also features displayed in this research study (West & Pines, 1985). Their description of intuitive knowledge as arising from interaction with the environment was seen in students' use of *everyday language* and their use of visually oriented language. In particular, students used continuity, slope, size and magnification as image schemata (Johnson, 1987). West and Pines' description of school knowledge as "someone else's interpretation of the world" or "someone else's reality" (p.3) was also seen in this study, in Collector students' views of their own learning. Further, West and Pines' use of a vine metaphor to represent states of the extent of intertwining of intuitive and school knowledge can also be applied to interpretation of students' calculus knowledge. Collectors and Technicians displayed intuitive knowledge that was not integrated with school knowledge. For example, conceptualizations built from *everyday language* knowledge as a *source of conviction* were not necessarily used in conjunction with conceptualizations built from knowledge of statements, rules and procedures. In comparison, Connectors displayed congruence between intuitive and school knowledge in that their school based knowledge was displayed as personally meaningful knowledge.

Connectors' integration of these two forms of knowledge can also be described as private interpretation and internalization of public knowledge (West et al., 1985). As a consequence Connectors' learning can be viewed as a process of giving personal, private meaning to public knowledge. In addition, since Ernest's (1991) notions of subjective and objective knowledge are similar to those of private and public knowledge, Connectors' learning can also be described as subjective interpretations of objective knowledge.

Some of the findings of this study in relation to students' calculus conceptualizations are congruent with previous research. First, students exhibited structural and executive errors similar to those of the interview students of Orton's studies (Orton, 1983a; Orton, 1983b). Structural errors displayed by students included misconceptions of limits, rounding off and continuity. Executive errors included a failure to correctly execute differentiation rules. Second, the findings of this study agree with those of Heid (1988) in how the nature and role of students' conceptualizations can be influenced by the nature of instruction. In particular, Gamma College students exhibited more use of *everyday language* than the other students. This feature, more *everyday language use*, was also evident in Gamma College instruction. As well, Beta College students exhibited more use of figures and more extensive description of figures than the other students, a feature that was evident in Beta College instruction. Lastly, Tall's (1989) report of students' use of the "cognitive root" or "anchoring concept" of local straightness was evident in Gamma College students' use of the features of an infinitely magnified portion of a graph.

Finally, this research study's findings also support previous research studies of student understandings of limits (Tall & Vinner, 1981; Davis & Vinner, 1986; Williams, 1991). Tall & Vinner (1981) found students' limit notions held both intuitive and formal ideas, with intuitive ideas explained in reference to sensory-motor processes. Davis and Vinner (1986) and Williams (1991) found students held a variety of limit misconceptions, in particular, that a limit cannot be reached. These findings were also findings of this research study.

## D. Implications for Instruction

In this section the three instructional approaches that formed the setting of this study are first reviewed and summarized. They were: technique-oriented, concepts-first, and infinitesimal instruction. Thus, the following points summarize the fifth area of inquiry of this study.

### The Three Instructional Approaches as Delivered to Students

The technique-oriented approach to instruction used at Alpha University is traditional in its emphasis on learning techniques for differentiation, integration, graphing and problem solving. Professor Alpha's presentations were organized, logical and mathematically elegant. He often presented ideas in a general form first, using concise, correct mathematical terminology. He then generally followed with specific examples. Professor Alpha spoke of student learning in calculus in ways that could be ascribed to a constructivist orientation. He said the way to learn calculus is "by doing", and he described students' calculus learning as a process of "confronting problems" so as to "learn to organize" one's thinking. In addition, he stated that a primary goal of introductory calculus is development of students' problem solving skills.

Concepts-first instruction, the instructional approach used at Beta College, explores concepts intuitively before introduction of formal definitions and proofs, and before skill development is emphasized. This instruction involves a spiral approach to topics in that concepts are revisited and developed at a more detailed and rigorous level as the school term progresses. Professor Beta's presentations varied in terms of the format followed and the level of rigor used to present ideas, with the level of rigor increasing throughout the term. However, as he introduced more formal presentations he continued to incorporate informal, graphical interpretations of concepts. Similarly to Professor Alpha, Professor Beta spoke of students' calculus learning in ways reminiscent of constructivism. He described calculus learning as involving "struggle", "hard thinking" and doing the work for oneself, and he remarked that he believes there are a variety of ways students learn concepts.

The infinitesimal approach to instruction used at Gamma College aims to develop concepts intuitively through the use of methods from nonstandard analysis. Key to this approach is the replacement of limits by rounding off, and the development of the derivative through the process of infinite magnification. Professor Gamma regularly incorporated group problem solving sessions into his classes, and his presentations were generally conducted in a questioning mode. He focused on talking about concepts and developing connections amongst symbols, words and graphs. Similarly to Professors Alpha and Beta, Professor Gamma spoke of calculus learning as a matter of doing the work for oneself. In addition, he saw learning how to think and solve problems as key objectives of introductory calculus.

The findings of the systematic classroom observations indicate that concepts-first instruction at Beta College as implemented by Professor Beta involved a higher percentage of physical context events (usually a graph) (PC) and physical experience as a *source of conviction* (IE) than did instruction at the other two institutions. In relation to this finding, interview students from Beta College exhibited more use of figures (usually graphs) and gave lengthier explanations of graphs in their problem responses than did the other students. Thus, it appears that Beta College students' more extensive use of figures (usually graphs) is likely due to the higher exposure to figures in classroom instruction at Beta College. Similarly, findings of the systemic classroom observations indicate infinitesimal instruction as implemented by Professor Gamma involved a higher percentage of use of spoken *everyday language* (EL(S)), as well as use of mathematical contexts (MC) or mathematics as a *source of conviction* (IM). In correspondence to these features is the fact that students at Gamma College displayed more use of ***everyday language*** than the other students and they exhibited a higher degree of appropriate integration of *everyday* and *technical language* (including symbols).

## The Impact of Instruction on Student Learning

The points outlined above under the summary of research objective (5) indicate that few definite conclusions can be made at this time as to the impact of instruction on students' *language use, sources of conviction,* and manner of construction of conceptualizations. First, it can be stated that infinitesimal instruction as implemented by

Professor Gamma encouraged students' use of *everyday language*, as well as appropriate integration of *technical* and *everyday language*. On a qualitative level these differences were manifested in some Gamma College students' concurrent use of infinitesimal *technical language* and *everyday language*. When students used infinitesimal language and the related notions of infinitesimal closeness and infinite magnification they were as tools by which to construct explanations and problem responses. The second conclusion that can be made at this time is that concepts-first instruction as implemented by Professor Beta enhanced students' abilities to examine and use graphical interpretations of calculus ideas.

The implications for instruction of these findings are twofold. First, when students used infinitesimal language and used it in conjunction with *everyday language* they generally did so as a foundation or *source of conviction* by which to construct problem responses. This finding indicates instruction that emphasizes connections between *everyday* and *technical language* is likely to guide students to build inter-connected conceptualizations. It is also likely to help students develop a sense of personal understanding of their calculus conceptualizations. Second, Gamma College students' use of infinite magnification in a variety of problem situations and Beta College students' use of graphs (figures) demonstrates that instruction emphasizing visual interpretations can impact upon students' conceptualizations. It can impact in that it can guide students to use bodily experiences as *sources of conviction*. The importance of these *sources of conviction* as revealed in this study is that they are means by which students construct and access calculus conceptualizations.

Since few of the students in this study were Connectors it is apparent that the search for effective ways to guide students to personal understandings of calculus must continue. Regardless of whether or not students apply calculus or study calculus beyond an introductory level, it is desirable that they pursue their calculus learning as a meaningful endeavour. What is noteworthy here is that students who saw their calculus learning as personally understandable displayed more competence, confidence and satisfaction in their abilities to do calculus.

On the basis of this study it appears that image schemata (Johnson, 1987) might serve as effective theory by which to develop and implement calculus instruction that guides students to meaningful calculus learning. Students' use of bodily experiences of continuity, slope, size or magnification support this claim. Their use of these bodily experiences as *sources of conviction* indicates students are able to construct calculus meanings from bodily experiences of the world. This feature in turn implies that instruction which emphasizes use of visual and physical calculus representations is likely to enhance students' sense of personal understanding of calculus. It therefore might also encourage students to construct their conceptualizations from internal *sources of conviction*. A point of the potential importance of internal *sources of conviction* as revealed in this study is that students who exhibited internal *sources of conviction* (Connectors) also generally had higher Completion Scores than other students, and displayed more competence, confidence and satisfaction in their abilities to do calculus.

Another implication for instruction that this study reveals is the important role that *language use* as a *source of conviction* plays in calculus learning. All students demonstrated their knowledge of calculus language was a factor in the nature of their calculus conceptualizations. Future efforts to make calculus instruction more successful for students must not neglect the role of *language use*. For example, Collector students used *everyday language* as a *source of conviction*, but often did not recognize related conceptualizations as valid mathematical interpretations. Increased instructional emphasis on the use of *everyday language* to construct conceptualizations and the integration of these conceptualizations with mathematically appropriate and precise conceptualizations might better guide these students to personal understandings of calculus. In a similar way, although Technicians used knowledge of calculus language as a technology by which to apply calculus, their related conceptualizations are not necessarily perceived by them to be personally meaningful. In other words, mastery of the use of calculus language can help students attain competence with calculus skills and basic ideas, but it does not necessarily guide them to personal understandings of their calculus conceptualizations. Thus, it appears that *language use* is an important vehicle by which calculus students might be better guided to calculus learning as a meaningful endeavour.

Related to a use of constructivist notions to guide instruction it must be noted that from this study it is clear many introductory calculus students do not perceive of their learning as personal, meaningful constructions. The fact that about half the interview students were classified as Collectors, who conceived of calculus learning as replication of teacher or textbook presentations, implies calculus instruction might be more successful for students if methods were developed that encourage students to take more personal involvement in the construction of their calculus conceptualizations.

At this point the possible use of radical constructivism in calculus instruction must be addressed. Radical constructivism sees learning as "organizing experience so as to deal with a real world that cannot itself be known" (Kilpatrick, 1987; p.6). Since this perspective is not in accordance with student's views of mathematics it is not immediately clear how radical constructivism might inform as to potentially beneficial avenues for future calculus instruction. Where radical constructivism might be a guide for future instruction is in the notion that mathematical objectivity is a social construction (Ernest, 1991). If instruction were designed to promote calculus learning as a process of subjective construction of publicly shared knowledge, then students might naturally be guided to build conceptualizations from internal *sources of conviction*. In particular, this sharing should include a mutual sharing and negotiation between teachers and students of use of symbols and *technical* and *everyday language* phrases, along with personal calculus interpretations of figures, statements and procedures.

## E. Emergent Themes and Recommendations for Further Research

This study investigated student learning in calculus from a constructivist perspective, using *language use* and *sources of conviction* as reflectors of the nature of student's calculus conceptualizations. A number of themes emerged during data analysis in this study, particularly in relation to the student interviews. Each theme is worthy of further investigation aimed at clarification, refinement and generalizability of notions. These themes will now be discussed.

(1) This study used *language use* and *sources of conviction* as tools by which to study students' calculus learning. Studies should be undertaken to improve and refine these ideas. For example, the notion of *language use* as a *source of conviction*, and the use of visually oriented language, image schemata, and procedural language need to be incorporated into research into students' calculus learning. As well, use of the systemic classroom observations and textbook analysis methods in research in other levels of mathematics learning would further determine the practicality and appropriateness of these methods for mathematics education research.

(2) A primary area of examination in this study was students' *sources of conviction*. This concept raises a number of issues in need of further research. First, studies should be undertaken to investigate whether the three groups of students, Collectors, Technicians and Connectors, are present in other groups of calculus students. Whether these groups are present in students studying mathematics at other levels or studying other subjects also needs to be determined. Such studies would contribute to the generalizability of this study, and would aid further application of constructivism to mathematics education and other areas of education.

(3) This study indicated a relationship between students' perceptions of learning calculus, and use of external or internal *sources of conviction*. Collectors displayed a lack of

confidence in calculus, perceived calculus as separate from their own reality, and approached their calculus learning by use of external *sources of conviction.* Technicians displayed a mastery of calculus as a technology, and they used a mixture of external and internal *sources of conviction.* Finally, Connectors exhibited the highest degree of internalization of calculus knowledge, and they generally displayed a higher level of proficiency with calculus ideas and skills. For these three groups what is not clear at this point is the nature of the relationship between various characteristics they exhibit. For example, it is not clear if Collector students' lack of confidence arises from a lack of personal understanding of calculus, or if lack of confidence causes one to use external *sources of conviction.* In a similar way, it is not clear if Technician students' use of calculus as a technology is inherent in the nature of their *sources of conviction*, or if the mixed external and internal nature of their *sources of conviction* encourages the use of calculus technology as a viable mode of functioning. It is also not clear if Connectors' relatively high level of competency in calculus arises from use of internal *sources of conviction*, or if competency leads to use of internal *sources of conviction.* Finally, research needs to be done to determine if the nature of Collectors', Technicians' and Connectors' approach to calculus learning forms a series of transitional learning phases. For example, it is not known if being a Technician might be a transitional phase between being a Collector and being a Connector.

(4) Another finding of this study, that *language use* is a *source of conviction*, is also in need of further examination. It is not presently clear if use of image schemata, visually oriented language or procedurally oriented language is external or internal in nature. In particular, points that are not clear at this point in relation to *language use* are the following:
(a) Students who are Connectors generally use symbols more extensively than Collectors or Technicians. Does this more extensive use of symbols give rise to approaching learning as a Connector, or does a student's approach to learning as a Connector foster facility with symbol use and connections between symbols, *technical language*, and *everyday language*.
(b) How might use of *everyday language* as a *source of conviction* be employed to guide students to construction of personally meaningful calculus conceptualizations? In particular, since Gamma College students displayed more appropriate integration of *technical* and *everyday language*, the influence of infinitesimal language on students' calculus conceptualizations needs further examination.
(c) The notion of *language use* as a *source of conviction* needs to be studied with students studying mathematics other than calculus. Not only would the generalizability of this study

be enhanced, it would extend the use of constructivist theory to mathematics education research at all levels of mathematics.

(d) Further research is also needed into how instruction can affect students' *language use* and *sources of conviction.* In particular, implementation studies of instruction designed to encourage students to construct conceptualizations of which they feel ownership are needed. Means by which this ownership might be brought about have already been discussed, in the previous section on educational implications. What also needs to be determined is how change in students' perceptions of mathematics, perceptions of mathematics learning, and approach to mathematics learning can be brought about.

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## Appendix A - Consent Form and Background Questionnaire

Department of Secondary Education
University of Alberta
Edmonton, Alberta
T6G 2G5

September 1990

Dear Student:

Your calculus class is part of a study into student learning in calculus. This study will be looking at what students learn in their calculus courses, and how their learning is related to various factors. No special knowledge or skills are needed as a prerequisite for being a participant in this study. Regular classroom instruction will not be disrupted, and participation in the study will not have anything to do with the determination of grades in this course. Confidentiality of results will be maintained at all times.

The results of this study will be valuable for the planning and implementation of instruction in undergraduate calculus courses. If you have any questions about this study, please contact me at the University of Alberta (telephone 492-3760), or speak to me after class one day.

Yours sincerely,

Sandra D. Fry

If you agree to be a participant in this study please sign below, then complete the questionnaire on the following pages.

| | |
|---|---|
| ______________________ | ______________________ |
| Name (please print) | Signature |

---

### Background Questionnaire for Calculus Students

The purpose of the following questionnaire is to obtain information on the educational background and interests of students enrolled in an introductory calculus course. All responses you put on the questionnaire will be confidential. Please answer all questions to the best of your ability. Some questions require a one or two word written response, but most require you to select from amongst a choice of answers. Mark your choice with an X.

If you did not attend high school in the province of Alberta, go to the next page.

Did you take Math 30? (if not, go to the next page)

Yes______
No______

How long ago did you complete Math 30?

within the last 12 months_______
between 1 and 3 years ago_______
between 3 and 5 years ago_______
more then 5 years ago_______

What mark did you receive on the Math 30 Diploma exam?

under 50%_______
50 - 59%_______
60 - 69%_______
70 - 79%_______
80 - 100%_______

What mark did you receive on your Math 30 term work?

under 50%_______
50 - 59%_______
60 - 69%_______
70 - 79%_______
80 - 100%_______

Did you ever repeat Math 30 or upgrade your mark?

Yes_______
No_______

Did you take Math 31?

Yes_______
No_______

Please go now to page 4.

If you did not attend high school in Alberta, in what province/state/country did you attend high school?

______________________________________________

If you did not attend high school in Alberta, or if you did not complete Math 30 in Alberta, what is the highest level mathematics course you took in high school?______________________________

How long ago did you complete this course?

within the last 12 months_______
between 1 and 3 years ago_______
between 3 and 5 years ago_______
more than 5 years ago_______

What mark (or equivalent) did you receive in this course?

under 50%_______
50 - 59%_______
60 - 69%_______
70 - 79%_______
80 - 100%_______

Did the course indicated above include the study of calculus?

Yes_______
No_______

Please proceed now to the next page.

The following questions are to be completed by all individuals.

Age:_______

Sex: male_______ female_______

Is English your first language?

Yes_______
No_______

If not, what is your native language?______________________________

If English is not your first language, indicate what English language training and experience you have:

English language training_________________________________________ Number of years living in an English speaking environment_______

In what type of community did you attend high school?

small rural community (under 3,000 people)_______
small town (3,000 - 10,000 people)_______
small city (10,000 - 50,000 people)_______
large city (over 50,000 people)_______

What grade levels were at this high school?

kindergarten - grade 12_______
grades 7 - 12_______
grades 10 - 12_______
other (please specify)_______

Approximately how many students were at this school?

fewer than 500_______
500 - 999_______
1000 - 2000_______
more than 2000_______

When did you last study any mathematics? (either at high school or college/university level)

within the last 12 months_______
between 1 and 3 years ago_______
between 3 and 5 years ago_______
more than 5 years ago_______

What is your major field of study ______________________________________

What is your minor field of study? (if applicable)______________________________

By course name, not number (eg. Canadian Politics), list all courses you will be taking this school term (Fall 1990).

If applicable, by course name (not number) list all university level math, science, or business courses you have taken in the past 2 years.

Is an undergraduate mathematics course required in your program of study?

Yes______
No______

Is calculus required in your program of study?

Yes______
No______

What is your career goal? (if known)____________________________________

Why did you choose to attend this college/university?

______________________________________________________________

Do you have any previous college or university diplomas or degrees?

Yes______
No______

If so, what diploma(s)/degree(s) do you have?_____________________________

On average, how many hours per day outside of class do you expect you will study calculus while you are enrolled in this course?

less than 1 hour______
between 1 and 2 hours______
between 2 and 3 hours______
more than 3 hours______

What final percentage grade do you expect to receive in this calculus course?

under 50%_______
50 - 59%_______
60 - 69%_______
70 - 79%_______
80 - 100%_______

How many times have you previously been enrolled in a college or university calculus course? (if you have never been enrolled before, proceed to the next page)

never enrolled before_______
once before_______
twice before_______
more than twice before_______

When did you last enroll in such a course?

within the last 12 months_______
between 1 and 3 years ago_______
between 3 and 5 years ago_______
greater than 5 years ago_______

Did you complete this course?

Yes_______
No_______

If so, what mark did you receive?

under 50%_______
50 - 59%_______
60 - 69%_______
70 - 79%_______
80 - 100%_______

If you did not complete this course, indicate why:

I withdrew during the first 2 weeks of the course. _______
I withdrew during the first half of the course because I wasn't doing well. _______
I withdrew during the second half of the course because I wasn't doing well. _______
I withdrew for other reasons (if possible, please indicate the reasons _______

___________________________________________________________

For each of the following statements, use an X to indicate whether you Strongly Agree (SA), Agree (A), are Uncertain (U), Disagree (D), or Strongly Disagree (SD):

| | SA | A | U | D | SD |
|---|---|---|---|---|---|
| I enjoy studying mathematics. | ____ | ____ | ____ | ____ | ____ |
| I often get confused by mathematical terminology and symbols. | ____ | ____ | ____ | ____ | ____ |
| Mathematics will be useful in my future career. | ____ | ____ | ____ | ____ | ____ |
| To do well in mathematics I have to work very hard. | ____ | ____ | ____ | ____ | ____ |
| I study mathematics because it is a required course in my program. | ____ | ____ | ____ | ____ | ____ |
| Generally, mathematics is a set of rules, formulas, and algorithms. | ____ | ____ | ____ | ____ | ____ |
| The thought of studying calculus makes me anxious. | ____ | ____ | ____ | ____ | ____ |
| Learning mathematics is mostly memorization and practice of certain problem types. | ____ | ____ | ____ | ____ | ____ |
| If I can get a math problem correct I don't worry about how or why things worked. | ____ | ____ | ____ | ____ | ____ |
| Mathematics is useful to society. | ____ | ____ | ____ | ____ | ____ |
| I expect to do well in this calculus course. | ____ | ____ | ____ | ____ | ____ |
| It is important to me to understand mathematics, not just to get right answers. | ____ | ____ | ____ | ____ | ____ |
| I find it easy to learn math if the teacher explains things in everyday language. | ____ | ____ | ____ | ____ | ____ |
| There is usually only one way to solve most mathematics problems. | ____ | ____ | ____ | ____ | ____ |

## Appendix B - End of Term Questionnaire

End of Term Questionnaire for Calculus Students

Dear Student:

The purpose of this questionnaire is to obtain information on the experiences of students enrolled in an introductory calculus course. All responses will be kept confidential. Please answer all questions to the best of your ability.

Name:________________________________

Sex: Male________ Female________

Was this course the first time you have taken a calculus course?
Yes_________ No_________

Before beginning this course were you apprehensive about studying calculus?
Yes________ No_________

Reasons (for either yes or no)________________________________
______________________________________________________

Has your attitude towards calculus changed since September?

Yes________ No________

If yes, in what way?_______________________________________
______________________________________________________

Are you exposed to calculus in any of your other courses?

Yes________ No________

Do you see calculus as useful to you in your future career?

Yes_________ No_________

Do you see calculus as useful to society?

Yes_________ No_________

Suppose someone were to ask you to briefly summarize what calculus is all about. What would you say to this individual?

______________________________________________________________
______________________________________________________________
______________________________________________________________
______________________________________________________________
______________________________________________________________

How well are you doing in the course at this point?
(please give an approximate percentage grade)__________________

How well are you doing in the course compared to what you had expected to achieve?

Higher__________ Lower__________ About the same__________

In an average week how much time do you spend on calculus outside of class?
(please give an estimate in hours)______________________________

How much time do you spend preparing for a major test or exam?
(please give an estimate in hours)______________________________

How is your workload in calculus compared with most of your other courses?

More work________ Less work________ About the same_______

Do you have sufficient time available to study and understand calculus?

Yes__________ No__________

Do you generally approach your calculus learning by recognition and memorization of problem types and solution methods?

Yes__________ No__________

Indicate which of the following you do at least once a week:
(check as many items as are applicable to you)

Read a section of the textbook before that section is covered in class. ______

Read a section of the textbook after that section is covered in class. ______

Use lecture notes to go over definitions and main ideas in order to understand them. ______

Work through on your own examples from the lecture notes or textbook. ______

Try to understand proofs or the derivation of concepts by reconstructing them for yourself. ______

Copy out or read through proofs or examples in order to try to memorize them. ______

Work on exercise questions that are additional to those assigned. ______

Work with other students on exercise questions. ______

Bring questions or difficulties to a lab/tutorial, or the instructor. ______

For each of the following statements, use an X to indicate whether you Strongly Agree (SA), Agree (A), are Uncertain (U), Disagree (D), or Strongly Disagree (SD):

| | SA | A | U | D | SD |
|---|---|---|---|---|---|
| I enjoy studying calculus. | ____ | ____ | ____ | ____ | ____ |
| I often get confused by calculus terminology and symbols. | ____ | ____ | ____ | ____ | ____ |
| I don't usually need the book or the teacher to tell me when I have done a question correctly. | ____ | ____ | ____ | ____ | ____ |
| Understanding main ideas and proofs is an important part of doing well in calculus. | ____ | ____ | ____ | ____ | ____ |
| I often have problems with the algebraic manipulation that is needed in calculus. | ____ | ____ | ____ | ____ | ____ |
| Generally, calculus is a set of rules, formulas, and algorithms. | ____ | ____ | ____ | ____ | ____ |
| I find it easy to learn calculus when the ideas are presented by pictures or graphs. | ____ | ____ | ____ | ____ | ____ |
| Learning calculus is mostly memorization and practice of certain problem types. | ____ | ____ | ____ | ____ | ____ |
| If I can get a calculus problem correct I don't worry about how or why things worked. | ____ | ____ | ____ | ____ | ____ |
| Calculus symbols and terminology are useful in learning calculus. | ____ | ____ | ____ | ____ | ____ |
| I can usually determine for myself the correctness of a calculus solution or proof. | ____ | ____ | ____ | ____ | ____ |
| It is important to me to understand calculus, not just to get right answers. | ____ | ____ | ____ | ____ | ____ |
| I find it easy to learn calculus if the teacher explains things in everyday language. | ____ | ____ | ____ | ____ | ____ |
| There is usually only one way to solve most calculus problems. | ____ | ____ | ____ | ____ | ____ |

If you have any additional comments about your calculus course please record them on the back of this page.

## Appendix C - Classroom Observation Summary Sheet

Site:____________________Date: __________Time: __________ Page: __________

| Time | Event | Language (Written) | Language (Spoken) | Convictions (Written) | Convictions (Spoken) | Other Observations |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |

## Appendix D - Sample Classroom Observation Notes and Corresponding Classroom Observation Summary Sheets

### A. Alpha University

(Monday, September 23)
Time
32

Example: Find $\lim_{x \to 0} f(x)$ if

$$f(x) = \begin{cases} 2x + 1, & x < 0 \\ 3x2 + 2, & x \geq 0 \end{cases}$$

*It is just one function defined in different ways on different parts of its domain.*

Solution

34

As $x$ gets close to zero from the right,

*That is, for positive numbers.*

$f(x) = 3x^2 + 2$ gets close to 2

*As $x$ gets close to zero from the left.*

$f(x) = 2x + 1$

36

*Now to make a limit statement . . .*
*Our alternative is to say the limit does not exist.*
*It's behaving pretty well on each side of zero.*
*We could say this function has a right-hand limit of 2 and a left-hand limit of one.*

38

Defn. We say the right hand limit of $f$ at $a$ is $L$ and write $\lim_{x \to 0^+} f(x) = L$ provided that as $x > a$ approaches $a$, $f(x)$ approaches $L$.

Likewise, left hand limit, $\lim_{x \to 0^-} f(x) = L$.

40

Example In the last example,

$$\lim_{x \to 0^+} f(x) = 2$$

$$\lim_{x \to 0^-} f(x) = 1$$

*The fact that these two limits differ means the limit does not exist.*

A. Alpha University Classroom Observation Summary Sheet for September 23:

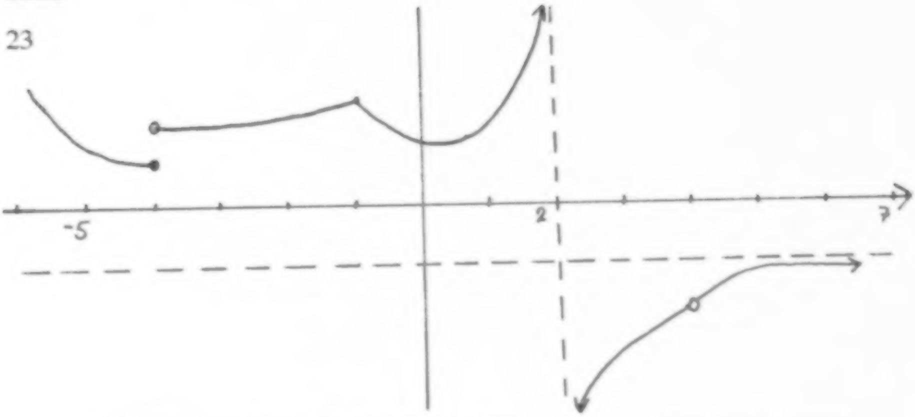
Site: Alpha Date: September 23 Time: 2 p.m. Page: 4

| Time | Event | Language (Written) | Language (Spoken) | Convictions (Written) | Convictions (Spoken) | Other Observations |
|---|---|---|---|---|---|---|
| 2:32 | EX<br>- spliced function | TL - limit<br>MC - limit evaluation | TL - symbols | IM - limit | | |
| 2:34 | EX<br>- solution | EL - gets close to | | IM-behaviour of function | | |
| 2:36 | CP<br>- two sided limit | MC-behaviour of function | EL- behaving well on each side<br>TL- rhl = 2 | | IM- behaviour of function | |
| 2:38 | CP-definition of one-sided | TL-terminology and symbols<br>MC-statement of definition | | ER-definition just stated | | |
| 2:40 | EX<br>-left and right hand limits | TL-symbols<br>MC-limits | | | IM-evaluation of limits | |

## B. Beta College

(Tuesday, October 16)
Time

23



*Few places where there's some sharp changes . . . gap in the graph.*
*Coming down here, then a gap.*
*Little circle with hole to indicate a point not on g.*

25

*Drew a little hole to indicate it's not defined there.*
*Gap in g.*
*Going to use this notion to examine it.*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*And you appreciate that that corresponds to slope.*

27

*Well the derivative is a limit.*

$$= \lim_{x \to -4} \frac{f(x) - f(-4)}{x - (-4)}$$

*Does that exist?*
*You're shaking your head no. Why?*
- (student responds)
*Does have a left-hand limit, but not right.*

does not exist since RHL does not exist

*. . . if I approach from the right . . .*
*. . . like f at minus 3 point nine nine.*

29

(2) f'(-1)

*Will that exist?*
*You have to think. We're not just taking the limit of a function, but an expression involving f. Looking at the slope.*
*Let's do it intuitively first.*

*Is there one tangent line you could draw at a?*
*Visualize taking the slope from the left and from the right.*

$$= \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} \text{ does not exist since LHL} \neq \text{RHL}$$

31

*Let's analyze a little more carefully.*

$$\text{LHL} = \lim_{x \to -1^-} \frac{f(x) - f(-1)}{x-(-1)} = \frac{\text{small negative}}{\text{small negative}} = +\text{ive}$$

*Take value of the function just to the left of minus one and subtract the value at negative 1.*
*Going to be positive or negative?*
- (student responds)
*Small negative . . . and what's going on in the denominator?*
*Think of it as minus one decimal zero one or something like that.*

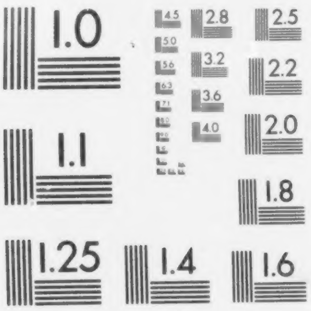
B. Beta College Classroom Observation Summary Sheet for October 16:

Site: Beta Date: October 16 Time: 1:15 p.m. Page: 2

| Time | Event | Language (Written) | Language (Spoken) | Convictions (Written) | Convictions (Spoken) | Other Observations |
|---|---|---|---|---|---|---|
| 1:23 | CP<br>-existence of derivative | PC-graph | EL-sharp changes<br>-gap<br>-little circle<br>-hole | | IE-graph | |
| 1:25 | CP<br>-as above | TL-symbols<br>PC-graph | EL-gap<br>-little hole | | IM-definition corresponds to slope | |
| 1:27 | CP<br>-as above | TL-symbols<br>PC-graph | EL-approach from the right | IM-left and right limits<br>IE-graph | | student response |
| 1:29 | CP<br>-as above | TL-symbols<br>PC-graph | EL-from the left and right | IM-RHL and LHL<br>IE-graph | | |
| 1:31 | CP<br>-analysis of left- hand limit | TL-symbols<br>EL-small negative<br>PC-graph<br>MC-left limit | EL-just to the left | IM-relative size and sign<br>IE-graph | | student response |

4

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT



## C. Gamma College

(Wednesday, October 3)
Time

(a graph of an arbitrary function has already been drawn and is referred to)

20

*. . . provided we agree on what we mean by close.*
*We can be more definite about close. . . . have hyperreals.*
*Can say infinitesimally close.*
*Really close.*

If we are infinitesimally close to some x = a then the corresp. y-values must be infinitesimally close to y(a).

*Lot more concise than drawing continuity . . . but still wordy.*
*How can I write a math expression that says . . . ?*
*Think about how we describe a change in x.*

22

Consider a + dx

This is infinitesimally close to a . . . because dx is an infinitesimal.

Note: dx can be + or -

Can you see how you would describe this notion now?

24

*What are the y-values that correspond to those?*

y's corresp. to x = a + dx are y(a + dx)

*If notation makes sense we can now use it.*

y at a + dx

*Are y-values corresponding to these x's.*
*What's y-value corresponding to x equal a?*
*What we need to have happen is what? y-values have to be close . . . this has to be close.*
*How can I write that down symbolically?*

26

*What's the connection of this to rounding off?*

If y(a + dx) ~> y(a)
for any + or - infinitesimal dx
then y is cont. at x = a.

28

Alt. for y = f(x) if f(a + dx) ~> f(a) for any + or - infinitesimal
then f is cont. at x = a

*- Is that d for delta?*

*How does dx relate to delta x?*
*Delta x is unspecified finite change in x versus . . .*

Gamma College Classroom Observation Summary Sheet for October 3:

Site: Gamma Date: October 3 Time: 8 a.m. Page: 3

| Time | Event | Language (Written) | Language (Spoken) | Convictions (Written) | Convictions (Spoken) | Other Observations |
|---|---|---|---|---|---|---|
| 8:20 | CP<br>-continuity | TL-infinitesimally close<br>PC-graph | EL-really close | IE-graph | IM-infinitesimally close | |
| 8:22 | CP<br>-as above | TL-symbols<br>MC-infinitesimally close<br>PC-graph | | | IM-use of infinitesimals | |
| 8:24 | CP<br>-as above | TL-symbols<br>PC-graph | EL-close to | IE-graph | IM-symbollic representation | |
| 8:26 | CP<br>-as above | TL-symbols<br>MC-round off | | | IM-rounding off | |
| 8:28 | CP<br>-as above | TL-symbols<br>MC-notation alternative | | | IM-dx versus delta x | student question |

# Appendix E - Summary Description of the Classroom Observation Summary Sheet Categories and Codes

Event

(1) (CP) Concept Presentation: The instructor develops or further explains concepts. This presentation might be in a general form, or might also be in conjunction with a specific example. Presentation of proofs is included in this category.
(2) (EX) Example: The instructor works through an example exercise problem to exemplify an idea, demonstrate a calculation, or solve a multistep problem.
(3) (O) Other: This includes administrative details such as collecting or handing back assignments, determination of test dates, or other events that are not explicitly instructional. Also included here are times when a class begins late or finishes early.

Language (Written or Spoken)

A. Language Type
(1) (TL) *Technical Language:* The language used is language generally accepted as proper and correct by the mathematics community at large.
(2) (EL) *Everyday Language:* The language used is not generally recognized by the mathematics community for use in unambiguous mathematical discourse. These words and symbols might or might not be mathematical in nature and are often words found in daily English language use.

B. Context
(1) (MC) Mathematical Context: The circumstances of the instruction are mathematical in nature and this is made explicit through the language used.
(2) (PC) Physical Context: The circumstances of the instruction refer to or use sensory-motor experiences of the world. Included here are graphs or diagrams, and mention of physical objects such as cars or hills.
(3) (CF) Context Free: The instruction is rule-governed, without reference to the origin of the rules.

Convictions (Written or Spoken)

(1) (IM) Internal/Mathematics: Truth and validity claims are made in reference to previously established mathematics, or through logical necessity.
(2) (IE) Internal/Experience: Truth and validity claims are made in reference to sensory-motor experiences. These references include use of graphs or diagrams, and reference to physical objects.
(3) (ER) External/Rules: A rule or rules are followed that either have not been previously justified or are not used with justification as to the choice of particular rules.
(4) (EO) External/Other: Truth and validity claims are made without any source being given, or the source acknowledged is the textbook, lab manual, or other document.

## Appendix F - Summary Description of the Textbook Analysis Summary Sheet Categories and Codes

Event

(1) (CP) Concept Presentation: The text material develops or further explains concepts. This might be in a general form, or in conjunction with a specific example. Presentation of proofs is included in this category.

(2) (EX) Example: The text material is an example exercise problem.

(3) (EXC) Exercise: The text material is an exercise for the student to work through on her or his own.

Language

A. Language Type

(1) (TL) *Technical Language:* The language used is language generally accepted as proper and correct by the mathematics community at large.

(2) (EL) *Everyday Language:* The language used is not generally recognized by the mathematics community for use in unambiguous mathematical discourse. These words and symbols might or might not be mathematical in nature and are often words found in daily English language use.

B. Context

(1) (MC) Mathematical Context: The language of the textbook event is explicitly mathematical in nature.

(2) (PC) Physical Context: The textbook event refers to or uses sensory-motor experiences of the world. Included here are graphs or diagrams, or mention of physical objects.

(3) (CF) Context Free: The textbook event states rules, ideas, or procedures without reference to their origin.

Sources of Conviction

(1) (IM) Internal-Mathematics: Truth and validity claims are made in reference to previously established mathematics, or through logical necessity.

(2) (IE) Internal-Experience: Truth and validity claims are made in reference to sensory-motor experiences. This includes use of graphs or diagrams, and reference to physical objects.

(3) (ER) External-Rules: A rule or rules are followed that either have not been previously justified, or are not used with justification as to the choice of the particular rule or rules.

Type

(See Appendix O)

## Appendix G - Instructor Interview Question Sheet

Teaching Background

1. How many years have you been teaching at this university/college?
2. What courses have you taught in that time? (type, level, size)
3. Did you have any teaching experience before this? If so, what?

Current Teaching

1. What is it like to teach in this department/college? What are the things you like and dislike about teaching?
2. What is your present teaching load? (courses, number of students, amount of preparation/grading time, etc.)
3. How much time do you make available for students outside class?
4. Do you find students come during your office hours? What types of questions do they bring?

Teaching Calculus

1. What factors does this department/college consider in planning the calculus program as it presently is? (content, nature and format of assignments, tests, exams) Who makes these decisions?
2. What do you see as the strengths/weaknesses of the Alberta high school math curriculum in terms of how it prepares students for learning calculus?
3. What do you see as the major strengths/weaknesses of the Math xxxxxxx course as it presently is? (content, structure, teaching approach)
4. What do you see as the strengths/weaknesses of the textbook used in Math xxxxxxx?

5.* Have you changed the way you teach since using this approach to calculus? Have these changes been effective? In what way?

6. If you could change the Math xxxxxxx course, what would you do? (additions, deletions, format)
7. What are the things that help/hinder teaching this Math xxxxxxx class? (class size, the curriculum, time, location, interaction with other instructors)

* This question was not asked of the instructor at Alpha University because it did not pertain to his situation.

Impressions about Teaching and Learning

1. How would you describe Math xxxxxxx students? What sorts of abilities do they have? What seems to motivate their learning? Do these aspects influence how you teach? In what ways?

2. How able and/or willing to be self-reliant in directing their own learning do Math xxxxxxx students seem to be? How do you see this as influencing their learning? What could an instructor do to enhance self-reliance in learning?

3. What are your ideas on how students learn? On effective ways to teach? (ideas on use of graphs, algebra, informal and formal presentations, language, reasoning processes or proofs, determination of validity or truth)

4. Do you teach differently to different groups of students? In what ways?

5. Students seem to have difficulty learning calculus. What are your thoughts on this?

6. What aspects of calculus do you see as easy/difficult for students to learn? Why? How do you handle teaching these things?

7. What do you see as the essential ideas and skills that student in your calculus class should take away with them at the end of the course?

8. What factors do you take into account when planning and carrying out your teaching? (emphases, strategies, changing plans)

## Appendix H - Clinical Interview Calculus Problems

1. A friend of yours who knows nothing about calculus is wondering what it is all about. What would you say to your friend to explain what calculus is all about?

2. For each of the following sequences of numbers, decide whether the sequence has a limit. If so, what is this number?

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \dots$$

$$3.9, \quad 3.99, \quad 3.999, \quad 3.9999, \quad 3.99999, \quad 3.999999, \dots$$

(Gamma College)
2. For each of the following sequences of numbers, decide whether the sequence rounds off to a particular number. If so, what is this number?

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \dots$$

$$3.9, \quad 3.99, \quad 3.999, \quad 3.9999, \quad 3.99999, \quad 3.999999, \dots$$

3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5}$$

(b) What does "limit" mean to you?

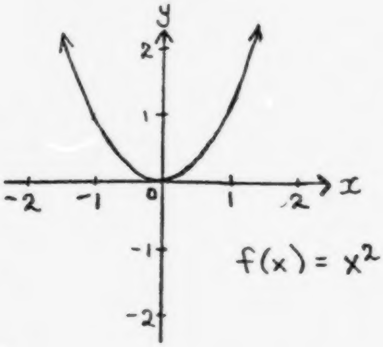
(Gamma College)
3. (a) Round off the following:

$$\frac{M^4 + 4}{M^3 - M + 5}$$

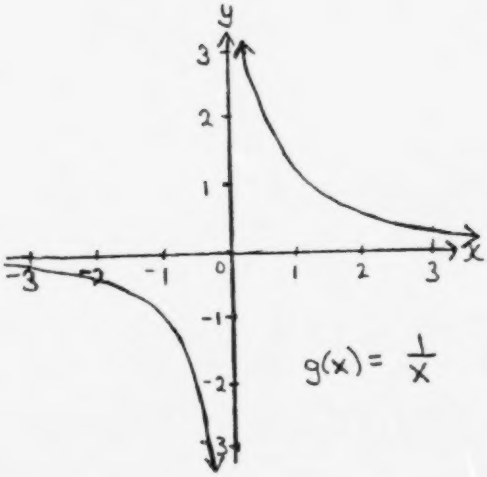
(b) What does "round off" mean to you?

4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?

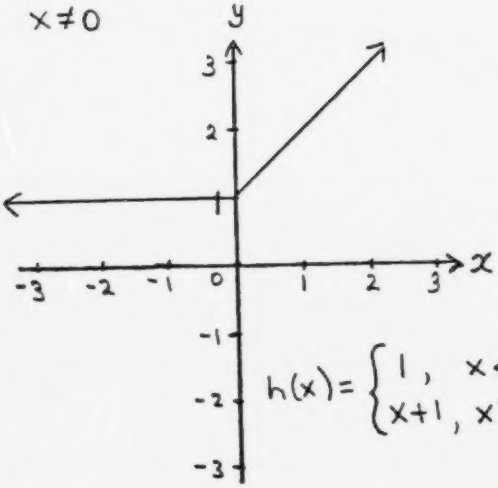
5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.



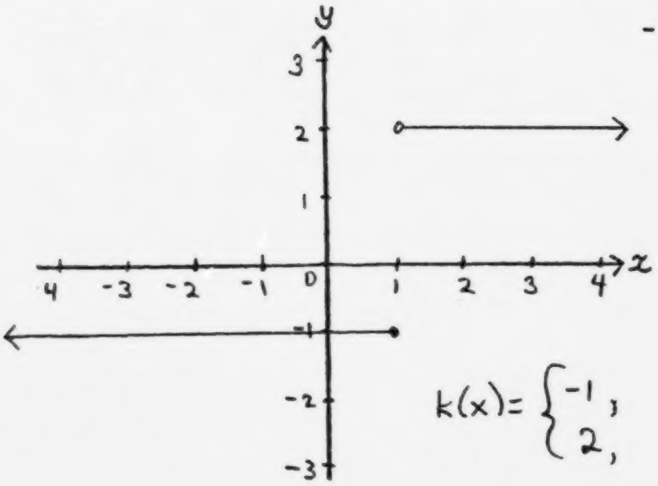
$f(x) = x^2$



$g(x) = \frac{1}{x} \quad x \neq 0$



$$h(x) = \begin{cases} 1, & x < 0 \\ x+1, & x \geq 0 \end{cases}$$



$$k(x) = \begin{cases} -1, & x \leq 1 \\ 2, & x > 1 \end{cases}$$

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative". What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?

7. Find the derivative of each of the following:

$$y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}$$

$$F(t) = (2t^2 + 3t - 2)^{10}(3t^{1/4} - 9)^7$$

8. What interpretations do you have for the expression below?

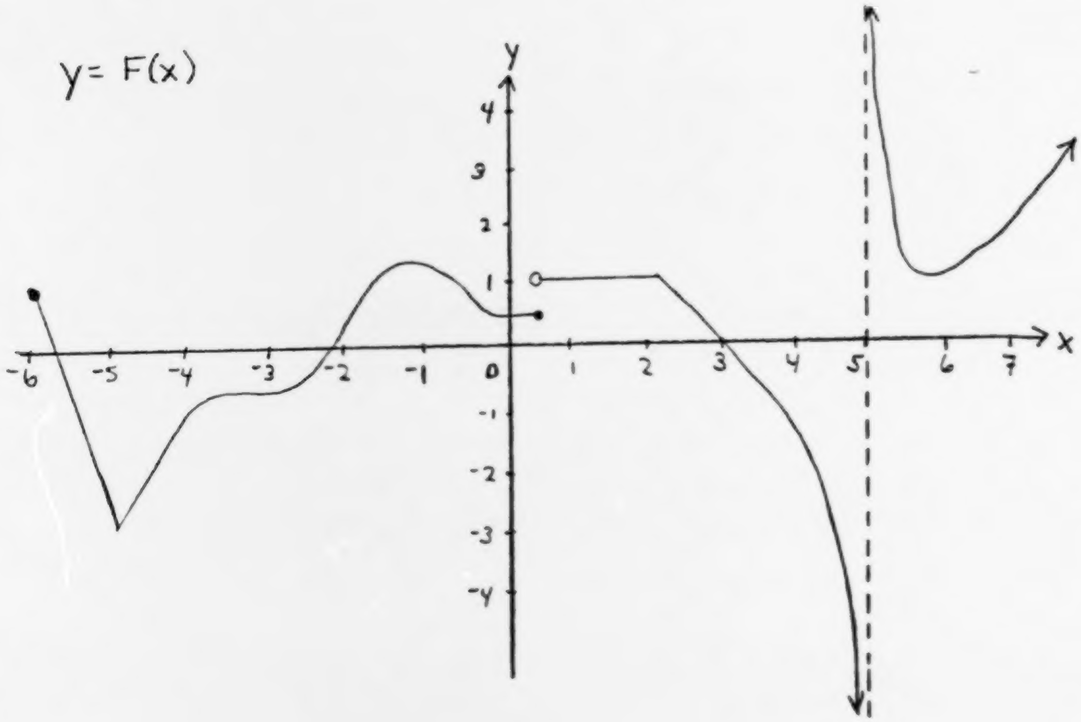
$$\lim_{x \to \infty} \frac{f(x+h) - f(x)}{h}$$

(Gamma College)

8. What interpretations do you have for the expression below?

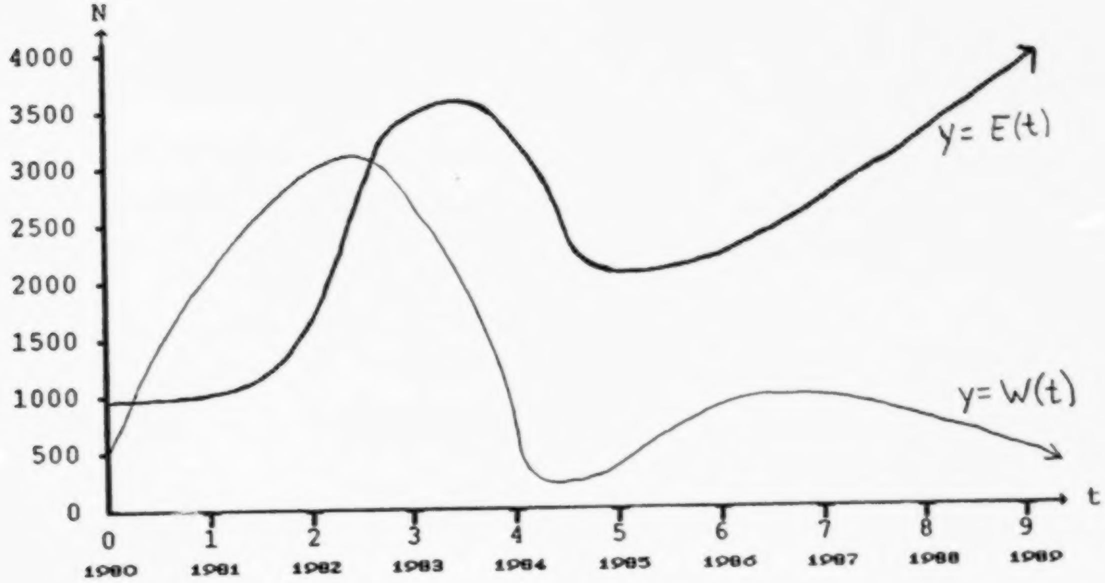
$$\frac{dy}{dx} = \frac{F(x+dx) - F(x)}{dx}$$

9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?



10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point $(0,-2)$.

11. The number of elk in a national park at the beginning of each year is represented by the function $y = E(t)$ as shown on the graph below. The number of wolves is represented by the function $y = W(t)$, also graphed below.



(a) At what exact point in time was the number of elk increasing most rapidly?

(b) During what time period was the rate of change of the number of elk decreasing?

(c) If you are told that for $0 < t < 4$ (ie. from 1980 to 1984) the equation for $y = g(t)$ is $W(t) = -100t^3 + 1600t + 500$ (t measured in years), how would you determine all critical points of W?

(d) How would you use the critical points found in part (c) to determine the local and global extrema of W?

(e) At what point or points in time is the number of wolves not changing?

12. On the axes given below, sketch the graph of a function with the following properties:

    (a) y coordinate of $-3$ when $x = -8$

    (b) derivative of $2$ when $x = -5$

    (c) local maximum when $x = -1$

    (d) derivative of $0$ when $x = 2$

    (e) slope of $1$ when $x = 4$

    (f) when $x = 7$, a point where the function is continuous but not differentiable

    (g) $f'(x) < 0$ and $f''(x) > 0$ when $x > 8$

## Appendix I - Sample Interview Probing Questions

(adapted from Clement and Konold, 1989)

1. Being an active listener:

   Could you please repeat what you just said?
   Slow down. I'm not following that quickly.

2. Encouraging vocalization:

   What are you thinking?
   Could you please explain what you are writing.

3. Asking for clarification/expansion:

   What do you mean?
   What more can you say about that?
   What other ways might you do/explain that?

4. Requesting reflection:

   How do you know that?
   How would you justify your answer to a fellow student who didn't believe you?
   Does that seem like a reasonable answer? Why?

# Appendix J - Student Personal Interview Questions

Student Interview Questions

1. What are your reasons for taking a calculus course?

2. Do you see calculus as useful to you? As useful to society?

3. Were you anxious or apprehensive about taking calculus before beginning this course in September? Why or why not?

4. Have your feelings about studying calculus changed since September? If so, in what way?

5. How well are you doing in the course at this point?

6. What are your thoughts on each of the following in terms of how they either help or hinder your calculus learning?

   (a) the textbook
   (b) assignment exercises
   (c) tests/quizzes
   (d) lectures
   (e) labs (if applicable)

7. How much time do you spend studying calculus? What do you do during that time?

8. What do you do when you encounter a difficulty while studying?

9. When you work with calculus ideas or solve calculus problems do you feel confident in what you are doing? Why or why not?

10. What does it mean to you to say you "understand" calculus?

11. Does the language (ie. symbols, terminology, descriptions) used in your calculus class or textbook help you to understand calculus? Why or why not?

12. What aspects of calculus do you find easy and what do you find difficult?

13. What things do you find help your learning in calculus? What things do you find hinder your learning?

14. Have you been exposed to calculus in any of your other courses?

## Appendix K - Selected Portions of the Textbook Tables of Contents

**A. Alpha University** (Stewart, J. Single Variable Calculus)

Chapter 1:
Numbers, Inequalities, and Absolute Values
The Cartesian Plane
Lines
Second-Degree Equations
Functions and Their Graphs
Combinations of Functions
Types of Functions and Transformed Functions

Appendix B: Review of Trigonometry

Chapter 2:
Tangents and Velocities
The Limit of a Function
Properties of Limits
Continuity

Chapter 3:
Derivatives
Differentiation Formulas
Derivatives of Trigonometric Functions
The Chain Rule
Implicit Differentiation; Angles Between Curves
Higher Derivatives
Rates of Change in the Natural and Social Sciences
Related Rates
Differentials

Chapter 4:
Maximum and Minimum Values
The Mean Value Theorem
Monotonic Functions and the First Derivative Test
Concavity and Points of Inflection
Limits at Infinity: Horizontal Asymptotes
Infinite Limits: Vertical Asymptotes
Curve Sketching
Applied Maximum and Minimum Problems
Antiderivatives

Chapter 5:
Sigma Notation
Area
The Definite Integral
Properties of the Definite Integral
The Fundamental Theorem of Calculus
The Substitution Rule
Areas Between Curves

## B. Beta College

Chapter 1:
The Derivative as Slope or Rate of Change
Differentiation Formulas
Elementary Maxima and Minima Problems
The Chain Rule; Implicit Differentiation
Derivatives of Trigonometric Functions
Related Rates

Chapter 2:
Introduction to the Limit
The Limit Notation
Some Rules for Evaluating Limits
Right and Left-hand Limits
Limits Yielding Infinity of 0/0
Limits where the Independent Variable goes to Infinity
Horizontal and Vertical Asymptotes
Two Trigonometric Limits

Chapter 3:
The Definition of the Derivative
Calculating Derivatives from the Definition
The Existence of the Derivative
Proof of Differentiation Rules

Chapter 4:
Introduction to Continuity
Continuity at a Point
Continuity on an Interval
Differentiability and Continuity

Chapter 5:
Absolute Value and Inequalities
Precise Definition of the Limit

Chapter 6:
Absolute maxima, minima
Local maxima, minima
Critical Points
Mean Value Theorem
First Derivative Test
The Second Derivative Test
Graphing
Applications of Maxima, Minima

Chapter 7:
Applications of differentials

Chapter 8:
Antiderivatives and indefinite integrals
A few areas using antiderivatives
Three problems involving sums
Application of the definite integral to area problems
Substitution, an aid in solving integrals
More general areas by integration

Chapter 9:
Summation Notation
Finding an area using summation
The Definite Integral
The Fundamental Theorem of Calculus
Prologue

### C. Gamma College

Chapter 1:
The Real Numbers
Analytic Geometry
The Line
Functions
Special Functions
The Inverse of a Function

Chapter 2:
The Hyperreal Number System
Rounding Off a Hyperreal Number
Functions of a Hyperreal Variable
Continuous Functions
Applications to Graphing
The Tangent Line

Chapter 3:
The Derivative
The Rate of Change of a Function
Derivative Formulas I
Derivative Formulas II
The Chain Rule
Implicit Differentiation
Higher Order Derivatives

Chapter 4:
The First Derivative and Graphing
The Second Derivative and Graphing
Maximum/Minimum Problems
Limits
Antiderivatives

Chapter 5:
The Sigma Notation
The Definite Integral
The Fundamental Theorem of Calculus
Change of Variable in Integrals
Properties of Definite Integrals

## Appendix L - Classroom Data Analysis Sheet

| | Time Interval | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Code | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | Total |
| CP | | | | | | | | | | | | | | | | | | | | | | | | | | |
| EX | | | | | | | | | | | | | | | | | | | | | | | | | | |
| O | | | | | | | | | | | | | | | | | | | | | | | | | | |
| MC | | | | | | | | | | | | | | | | | | | | | | | | | | |
| PC | | | | | | | | | | | | | | | | | | | | | | | | | | |
| CF | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TL(w) | | | | | | | | | | | | | | | | | | | | | | | | | | |
| TL(s) | | | | | | | | | | | | | | | | | | | | | | | | | | |
| EL(w) | | | | | | | | | | | | | | | | | | | | | | | | | | |
| EL(s) | | | | | | | | | | | | | | | | | | | | | | | | | | |
| IM | | | | | | | | | | | | | | | | | | | | | | | | | | |
| IE | | | | | | | | | | | | | | | | | | | | | | | | | | |
| ER | | | | | | | | | | | | | | | | | | | | | | | | | | |
| EO | | | | | | | | | | | | | | | | | | | | | | | | | | |

# Appendix M - Sample Completed Classroom Data Analysis Sheet

| | Time Interval | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Code** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** | **21** | **22** | **23** | **24** | **25** | **Total** |
| **CP** | ○ | ○ | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 5 |
| **EX** | | | | | | | | | | | | | | | | | | | | | | | | | | 0 |
| **O** | | | | | | | | | | | | | | | | | | | | | | | | | | 0 |
| **MC** | | | | | ○ | | | | | | | | | | | | | | | | | | | | | 1 |
| **PC** | ○ | ○ | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 5 |
| **CF** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **TL(w)** | | ○ | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 4 |
| **TL(s)** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **EL(w)** | | | | | ○ | | | | | | | | | | | | | | | | | | | | | 1 |
| **EL(s)** | ○ | ○ | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 5 |
| **IM** | | ○ | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 4 |
| **IE** | ○ | | ○ | ○ | ○ | | | | | | | | | | | | | | | | | | | | | 4 |
| **ER** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **EO** | | | | | | | | | | | | | | | | | | | | | | | | | | |

## Appendix N - Textbook Analysis Summary Sheet

| | |
|---|---|
| Type | |
| Sources of Conviction | |
| Language | |
| Event | |

## Appendix O - Summary Description of the Textbook Example and Exercise Codes

Examples (EX)

A. (I) Imitation

Students could duplicate the steps in the example with a variety of exercise questions similar to the example.

(1) (d) demonstration: demonstration of a type of calculation or procedure, or application of a rule.
(2) (p) property: a specified property is displayed through a graph, equation, or numerical or algebraic expression.
(3) (w) word problem: a one or two step application of a concept or procedure. This application is to a physical context, as opposed to application to another area of mathematics.

B. (N) Non-Imitation

Students are not likely to be able to duplicate the steps in the example with a variety of exercises similar to the example.

(1) (m) multistep: an application (within either a mathematical or physical context) of concepts or procedures that involve one or more of the following in reaching a solution: analysis of a situation, synthesis of several concepts, or construction of a graph, equation or expression.
(2) (i) interpretion: interpretation or explanation of a graph or mathematical or physical situation.

(v) Visual Component : a graph or diagram is present in the example.

Exercises (EXC):

A. (R) Routine

Tasks for which a procedure leading to a solution has been presented in the textbook.

(1) (i) identification: identification or recognition of a property or concept.

(2) (a) algorithm: use of a rule or algorithm.

(3) (w) word problem: a one or two step application of a concept or procedure. This application is to a physical context, as opposed to application to another area of mathematics.

B. (T) Transitory

Tasks for which procedures leading to a solution have been presented in the text, but the solution procedures involve several steps, or interpretion of notation or graphs.

(1) (g) graphing: application of rules along with graphing of the results.

(2) (a+) application: use of several rules or algorithms, or use of a rule or algorithm that involves interpretation of notation or interpretation of a graph.

C. (P) Problem (P)

Tasks for which a procedure leading to a solution is not known.

(1) (m) multistep: a task involving more than one of the following: identification of a property or concept, analysis of a situation, synthesis of concepts or calculation results, application of rules or algorithms, derivation of an equation or formula, or sketching of a graph.

(2) (c) create: create an example of a situation, function or equation that possesses specified properties (graphs are not included here).

(3) (cg) construct a graph: construct a graph which possesses specified properties.

(4) (p) prove: prove a general result.

(5) (e) explain: explain, describe or interpret a mathematical situation. Graphs are not included here.

(6) (ig) interpret a graph: interpretation of a graph.

## Appendix P - Sample Completed Textbook Analysis Summary Sheet

| Event | Language | Sources of Conviction | Type |
|---|---|---|---|
| CP-motivation for the derivative | TL-symbols, terms<br>MC, PC | IE, IM | |
| CP-definition of the derivative | TL-symbols, terms<br>EL, MC, PC | IE, IM | |
| EX-$x^2 + 2$ | TL-symbols MC | IM | Id |
| CP-F'(x) | TL-symbols<br>EL, MC | IM | |
| CP-slope of curve | TL-symbols, terms MC | ER | |
| EX-$x^2$ at $x = 1$ | TL-symbols, terms<br>MC | IM | Id |
| CP-differentiable | TL-symbols, terms<br>MC | ER | |
| EX-$x^2$ | TL-symbols, terms<br>MC | IM | Id |
| EX-cube root x | TL-symbols, terms<br>MC | IM | Id |
| EX-two graphs | TL-symbols, terms<br>EL, MC, PC | IE, IM | Id |

## Appendix Q - Textbook Data Analysis Sheet

| | Event Number | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Code** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | **Total** |
| **CP** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **EX** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **EXC** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **MC** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **PC** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **CF** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **TL** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **EL** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **IM** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **IE** | | | | | | | | | | | | | | | | | | | | | | | | | | |
| **ER** | | | | | | | | | | | | | | | | | | | | | | | | | | |

# Appendix R - Sample Language List

**Language List: Betty**

Everyday Language

Problem 2:
- getting smaller and smaller
- going forever
- bigger and bigger
- getting closer and closer to four; but not reaching four

Problem 3a:
- have to work it out so it's not in the indeterminate form
- can just drop these because they're infinitesimal and that would be the rounded off form
- can just drop it
- it's big

Problem 3b:
- start off with different things
- make calculations easier
- try to get them to look like whole numbers

Problem 4:
- curved line
- have to work it out and round off

Problem 5:
- two lines separated
- should be an open type interval, open point
- there's an open space and it doesn't say equal down here
- can draw without taking your pencil off the graph
- have to lift your pencil from one place to get to another place
- no breaks
- like a flow
- dots would be going up consecutively
- check the points
- so it will come down and go up continuously
- not connecting

Problem 6:
- see how it behaves, looks

Problem 7:
- first times derivative of second. ... first times the second.
- the inside

Problem 8:
- just put it down at the beginning

Problem 10:
- change over change

Problem 11:
- fastest it increases
- highest amount of increase; higher increase within a shorter amount of time
- pretty good drop going down
- increase the graph and focus on that flat line as you get closer, and then flatter and flatter
- slowly and slowly curve up
- almost the straightest line on the graph
- just varies a little bit
- there's a drastic change and then there's sort of a slight but not too much
- staying at that level
- sort of gets flat
- reaches a plateau
- sort of a stop, not really low or high

Problem 12:
- has to be big
- has to be going up

Technical Language (symbols used are on a separate list)

Problem 3a:
- indeterminate form
- infinitesimal dx
- zero on the bottom

Problem 4:
- undefined
- indeterminant

Problem 5:
- split function

Problem 6:
- the slope of the graph
- asymptote

Problem 7:
- quotient rule and power rule

Problem 9:
- endpoint
- no slope
- tangent line

Problem 10:
- y intercept

Problem 11:
- decreases

Problem 12:
- y less than zero
- slope greater than zero

Symbols

Problem 2:
- $\sim> \frac{1}{\infty}$

Problem 3a:
- $\frac{\infty}{\infty}$
- dx

Problem 3b:
- $\frac{dx + 1}{dx + 2}$

Problem 10:
- $\frac{\Delta y}{\Delta x}$
- $y = mx + b$

Problem 12:
- $y = mx + b$, $\frac{y}{x} = 1$
- $F(x) = 2$

## Appendix S - Outline Language Chart and Individual Student Language Charts

**Outline Chart:**

| | Context Category | | | | | |
|---|---|---|---|---|---|---|
| **Problem** | **TL-S** | **TL-W** | **EL** | **F** | **O** | **Completion Score** |
| 2 | ● | ● | | | | |
| 3a | ● | | | | | |
| 3b | | ● | | | | |
| 4 | ● | | | | | |
| 5 | ● | ● | | ● | | |
| 6 | | ● | | | | |
| 7 | ● | ● | | | | |
| 8 | ● | | | | | |
| 9 | | ● | | ● | | |
| 10 | ● | ● | | | | |
| 11 | | ● | | ● | ● | |
| 12 | ● | ● | | | | |

**Student:** *Annabel*

| Problem | Context Category | | | | | | | | Completion Score |
|---|---|---|---|---|---|---|---|---|---|
| | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | |
| 2 | | | | | ✔ | 1 | | | 3 |
| 3a | | | ✔ | 2 | ✔ | 2 | | | 3 |
| 3b | ✔ | 1 | ✔ | 1 | ✔ | 3 | ✔ | | 2 |
| 4 | | | ✔ | 2 | | | ✔ | | 2 |
| 5 | ✔ | 1 | ✔ | 4 | ✔ | 3 | | | 2 |
| 6 | ✔ | 1 | ✔ | 6 | ✔ | 1 | ✔ | | 2 |
| 7 | ✔ | 1 | | | ✔ | 1 | | | 3 |
| 8* | ✔ | | ✔ | | ✔ | | ✔ | | 3 |
| 9 | ✔ | 1 | ✔ | 5 | ✔ | 1 | ✔ | | 3 |
| 10 | | | ✔ | 1 | | | ✔ | | 3 |
| 11 | ✔ | 2 | ✔ | 6 | ✔ | 4 | | | 2 |
| 12 | | | ✔ | 3 | | | ✔ | | 2 |
| **Total** | 7 | 7 | 10 | 30 | 9 | 16 | 7 | 0 | 30 |

**Student:** *Ellen*

| **Problem** | **Context Category** | | | | | | | | **Completion Score** |
|---|---|---|---|---|---|---|---|---|---|
| | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | |
| 2 | | | ✔ | 1 | ✔ | 4 | | | 2 |
| 3a | | | ✔ | 1 | | | | | 2 |
| 3b | | | | | ✔ | 4 | | | 1 |
| 4 | ✔ | 1 | | | ✔ | 1 | ✔ | | 1 |
| 5 | | | ✔ | 1 | ✔ | 6 | | | 2 |
| 6 | | | | | | | | ✔ | 1 |
| 7 | ✔ | 1 | ✔ | 2 | ✔ | 2 | | | 2 |
| 8 | | | ✔ | 1 | | | | | 1 |
| 9 | | | ✔ | 1 | | | | | 1 |
| 10 | | | ✔ | 1 | | | | | 2 |
| 11 | | | | | ✔ | 4 | | | 1 |
| 12 | | | | | ✔ | 3 | | | 1 |
| **Total** | 2 | 2 | 7 | 8 | 7 | 24 | 1 | 1 | 17 |

**Student:** *Jennifer*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | ✔ | 1 | ✔ | 5 | ✔ | | 3 |
| 3a | | | | | | | | | 2 |
| 3b | | | ✔ | 2 | ✔ | 1 | | | 2 |
| 4 | | | ✔ | 2 | | | | | 2 |
| 5 | ✔ | 1 | ✔ | 2 | ✔ | 7 | | ✔ | 2 |
| 6 | ✔ | 1 | ✔ | 2 | ✔ | 2 | ✔ | | 2 |
| 7 | | | ✔ | 3 | ✔ | 1 | | | 2 |
| 8 | | | ✔ | 2 | ✔ | 1 | ✔ | | 2 |
| 9 | | | ✔ | 3 | ✔ | 7 | | | 2 |
| 10 | | | ✔ | 1 | | | | | 3 |
| 11 | | | ✔ | 1 | ✔ | 6 | | | 2 |
| 12 | | | ✔ | 3 | | | ✔ | | 2 |
| **Total** | 2 | 2 | 11 | 22 | 8 | 30 | 4 | 1 | 26 |

**Student:** *Ned*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | | | ✓ | 6 | | | 3 |
| 3a | | | ✓ | 2 | | | | | 3 |
| 3b | | | | | ✓ | 3 | | ✓ | 1 |
| 4 | | | ✓ | 2 | ✓ | 3 | ✓ | | 1 |
| 5 | | | | | ✓ | 12 | | | 2 |
| 6 | ✓ | 3 | ✓ | 5 | ✓ | 3 | ✓ | ✓ | 2 |
| 7 | | | ✓ | 1 | ✓ | 1 | | | 2 |
| 8 | | | ✓ | 2 | | | | | 0 |
| 9 | | | ✓ | 7 | ✓ | 7 | | | 2 |
| 10 | | | ✓ | 1 | | | | | 3 |
| 11 | | | | | ✓ | 1 | | | 2 |
| 12 | | | ✓ | 4 | ✓ | 1 | ✓ | | 2 |
| **Total** | 1 | 3 | 8 | 24 | 9 | 37 | 3 | 2 | 25 |

**Student:** *Richard*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✓ | 1 | ✓ | 1 | ✓ | 2 | | | 3 |
| 3a | | | ✓ | 2 | | | | | 3 |
| 3b | | | ✓ | 2 | ✓ | 5 | ✓ | | 3 |
| 4 | | | ✓ | 4 | ✓ | 2 | ✓ | | 2 |
| 5 | ✓ | 1 | ✓ | 2 | ✓ | 4 | | | 2 |
| 6 | ✓ | 2 | ✓ | 5 | ✓ | 2 | ✓ | ✓ | 3 |
| 7 | | | ✓ | 1 | ✓ | 2 | | | 3 |
| 8* | ✓ | | ✓ | | ✓ | | ✓ | ✓ | 3 |
| 9 | | | ✓ | 4 | ✓ | 2 | | | 3 |
| 10 | | | ✓ | 2 | | | ✓ | | 2 |
| 11 | | | ✓ | 4 | ✓ | 3 | | | 2 |
| 12 | | | ✓ | 2 | | | ✓ | | 2 |
| **Total** | 4 | 4 | 12 | 29 | 9 | 22 | 6 | 2 | 31 |

**Student:** *Cindy*

| Problem | Context Category | | | | | | | | Completion Score |
|---|---|---|---|---|---|---|---|---|---|
| | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | |
| 2 | | | ✓ | 2 | ✓ | 6 | ✓ | | 3 |
| 3a | | | | | ✓ | 6 | | | 3 |
| 3b | | | ✓ | 1 | ✓ | 2 | ✓ | | 2 |
| 4 | | | ✓ | 2 | ✓ | 1 | ✓ | | 1 |
| 5 | | | ✓ | 2 | ✓ | 2 | | ✓ | 2 |
| 6 | ✓ | 1 | ✓ | 1 | ✓ | 2 | ✓ | | 2 |
| 7 | | | ✓ | 2 | ✓ | 1 | | | 2 |
| 8 | ✓ | 1 | ✓ | 1 | | | ✓ | | 2 |
| 9 | | | ✓ | 2 | ✓ | 2 | | | 2 |
| 10 | ✓ | 2 | ✓ | 2 | | | ✓ | | 0 |
| 11 | | | ✓ | 5 | ✓ | 2 | | ✓ | 2 |
| 12 | | | ✓ | 6 | ✓ | 1 | ✓ | | 2 |
| **Total** | 3 | 4 | 11 | 26 | 10 | 25 | 7 | 2 | 23 |

**Student:** *Daniel*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | ✔ | 3 | ✔ | 5 | | | 1 |
| 3a | ✔ | 1 | ✔ | 3 | | | ✔ | | 1 |
| 3b | ✔ | 1 | ✔ | 1 | ✔ | 3 | ✔ | ✔ | 2 |
| 4 | | | ✔ | 4 | ✔ | 2 | ✔ | | 3 |
| 5 | | | ✔ | 3 | ✔ | 9 | | | 2 |
| 6 | ✔ | 1 | ✔ | 1 | ✔ | 3 | | ✔ | 1 |
| 7 | | | | | ✔ | 1 | | | 0 |
| 8 | | | ✔ | 1 | ✔ | 1 | | ✔ | 0 |
| 9 | | | ✔ | 4 | ✔ | 6 | | | 0 |
| 10 | | | ✔ | 2 | | | ✔ | | 0 |
| 11 | | | ✔ | 2 | ✔ | 1 | | | 2 |
| 12 | | | | | | | ✔ | | |
| **Total** | 3 | 3 | 10 | 24 | 9 | 31 | 5 | 3 | 14 |

**Student:** *Doug*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | ✓ | 1 | ✓ | 3 | | | 0 |
| 3a | ✓ | 1 | ✓ | 2 | ✓ | 5 | | ✓ | 2 |
| 3b | | | | | ✓ | 2 | | | 1 |
| 4 | | | ✓ | 2 | ✓ | 2 | ✓ | | 1 |
| 5 | | | ✓ | 2 | ✓ | 9 | | | 2 |
| 6 | | | ✓ | 1 | ✓ | 2 | | | 1 |
| 7 | ✓ | 1 | ✓ | 1 | ✓ | 3 | | | 2 |
| 8 | | | ✓ | 1 | | | | | 1 |
| 9 | | | ✓ | 8 | ✓ | 5 | | | 2 |
| 10 | ✓ | 1 | ✓ | 1 | | | | | 0 |
| 11 | | | ✓ | 4 | ✓ | 5 | | | 2 |
| 12 | | | | | | | ✓ | | 1 |
| **Total** | 3 | 3 | 10 | 23 | 9 | 36 | 2 | 1 | 15 |

**Student:** *Leanne*

| | **Context Category** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | ✔ | 1 | ✔ | 4 | ✔ | | 3 |
| 3a | ✔ | 2 | ✔ | 2 | ✔ | 1 | | | 3 |
| 3b | | | ✔ | 1 | ✔ | 5 | ✔ | | 2 |
| 4 | | | ✔ | 1 | ✔ | 2 | ✔ | | 2 |
| 5 | | | ✔ | 5 | ✔ | 1 | | | 2 |
| 6 | ✔ | 2 | ✔ | 4 | ✔ | 2 | ✔ | | 2 |
| 7 | | | ✔ | 1 | ✔ | 1 | | | 1 |
| 8 | | | ✔ | 1 | ✔ | 1 | ✔ | | 3 |
| 9 | | | ✔ | 10 | ✔ | 5 | ✔ | | 3 |
| 10 | | | ✔ | 1 | | | ✔ | | 1 |
| 11 | | | ✔ | 2 | ✔ | 3 | | | 2 |
| 12 | | | ✔ | 1 | | | ✔ | | 1 |
| **Total** | 2 | 4 | 12 | 30 | 10 | 25 | 8 | 0 | 25 |

**Student:** *Sally*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | | | ✔ | 5 | | ✔ | 3 |
| 3a | 4 | 1 | ✔ | 1 | ✔ | 2 | | | 3 |
| 3b | | | | | ✔ | 4 | ✔ | ✔ | 2 |
| 4 | | | ✔ | 5 | ✔ | 2 | ✔ | | 2 |
| 5 | | | ✔ | 2 | ✔ | 6 | | ✔ | 3 |
| 6 | 4 | 2 | ✔ | 3 | ✔ | 1 | ✔ | | 2 |
| 7 | 4 | 1 | ✔ | 2 | ✔ | 3 | | | 3 |
| 8* | 4 | | ✔ | | ✔ | | ✔ | | 3 |
| 9 | | | ✔ | 6 | ✔ | 6 | | | 3 |
| 10 | 4 | 1 | ✔ | 1 | | | ✔ | | 0 |
| 11 | | | ✔ | 6 | ✔ | 2 | | | 2 |
| 12 | | | ✔ | 4 | ✔ | 1 | ✔ | | 3 |
| **Total** | 5 | 5 | 10 | 30 | 11 | 32 | 6 | 3 | 29 |

**Student:** *Tim*

| Problem | Context Category<br>TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | Completion Score |
|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | ✓ | 1 | | | 3 |
| 3a | | 4 | ➡ | | ✓ | 2 | | | 3 |
| 3b | ✓ | 1 | ✓ | 1 | ✓ | 3 | ✓ | | 2 |
| 4 | | | ✓ | 2 | | | ✓ | | 2 |
| 5 | ✓ | 1 | ✓ | 4 | ✓ | 3 | | | 2 |
| 6 | ✓ | 1 | ✓ | 6 | ✓ | 1 | ✓ | | 2 |
| 7 | ✓ | 1 | | | ✓ | 1 | | | 3 |
| 8* | ✓ | | ✓ | | ✓ | | ✓ | | 3 |
| 9 | ✓ | 1 | ✓ | 5 | ✓ | 1 | ✓ | | 3 |
| 10 | | | ✓ | 1 | | | ✓ | | 3 |
| 11 | ✓ | 2 | ✓ | 6 | ✓ | 4 | | | 2 |
| 12 | | | 4 | 3 | | | ✓ | | 2 |
| **Total** | 7 | 7 | 10 | 30 | 9 | 16 | 7 | 0 | 30 |

**Student:** *Betty*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✔ | 1 | | | ✔ | 4 | | | 3 |
| 3a | ✔ | 2 | ✔ | 3 | ✔ | 4 | | | 2 |
| 3b | ✔ | 1 | | | ✔ | 3 | | | 1 |
| 4 | | | ✔ | 2 | ✔ | 2 | ✔ | | 1 |
| 5 | | | ✔ | 1 | ✔ | 11 | | ✔ | 2 |
| 6 | | | ✔ | 2 | ✔ | 1 | ✔ | | 1 |
| 7 | | | ✔ | 1 | ✔ | 2 | | | 2 |
| 8 | | | | | ✔ | 1 | | | 1 |
| 9 | | | ✔ | 3 | | | | | 1 |
| 10 | ✔ | 2 | ✔ | 1 | ✔ | 1 | ✔ | | 0 |
| 11 | | | ✔ | 1 | ✔ | 12 | | | 2 |
| 12 | ✔ | 2 | ✔ | 2 | ✔ | 2 | ✔ | | 1 |
| **Total** | 5 | 8 | 9 | 16 | 11 | 43 | 4 | 1 | 17 |

**Student:** *Gordon*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | | | | | ✔ | 5 | | | 3 |
| 3a | | | ✔ | 2 | ✔ | 2 | | | 3 |
| 3b | | | | | ✔ | 4 | ✔ | | 1 |
| 4 | | | ✔ | 2 | ✔ | 2 | ✔ | | 2 |
| 5 | | | ✔ | 1 | ✔ | 6 | | | 2 |
| 6 | ✔ | 1 | ✔ | 3 | | | ✔ | ✔ | 2 |
| 7 | | | | | ✔ | 1 | | | 1 |
| 8 | | | ✔ | 2 | ✔ | 1 | ✔ | | 1 |
| 9 | | | ✔ | 1 | ✔ | 6 | | | 2 |
| 10 | | | ✔ | 1 | | | | | 0 |
| 11 | | | ✔ | 5 | ✔ | 2 | | | 2 |
| 12 | | | ✔ | 1 | | | ✔ | | 2 |
| **Total** | 1 | 1 | 9 | 18 | 8 | 29 | 5 | 1 | 21 |

**Student:** *Mike*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✔ | 1 | ✔ | 1 | ✔ | 5 | | | 3 |
| 3a | ✔ | 1 | ✔ | 3 | ✔ | 4 | | | 3 |
| 3b | | | ✔ | 1 | ✔ | 4 | | | 2 |
| 4 | | | ✔ | 4 | ✔ | 3 | ✔ | | 1 |
| 5 | ✔ | 1 | ✔ | 2 | ✔ | 5 | | ✔ | 3 |
| 6 | ✔ | 3 | ✔ | 3 | ✔ | 4 | ✔ | ✔ | 3 |
| 7 | | | ✔ | 1 | | | | | 3 |
| 8 | ✔ | | ✔ | | ✔ | | ✔ | ✔ | 3 |
| 9 | ✔ | 1 | ✔ | 1 | ✔ | 13 | | | 3 |
| 10 | | | ✔ | 1 | | | | | 3 |
| 11 | | | ✔ | 5 | ✔ | 3 | | | 2 |
| 12 | | | ✔ | 1 | | | ✔ | | 2 |
| **Total** | 6 | 7 | 12 | 23 | 9 | 41 | 4 | 2 | 31 |

**Student:** *Nadine*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✔ | 1 | | | ✔ | 4 | | | 3 |
| 3a | | | ✔ | 3 | ✔ | 1 | | | 2 |
| 3b | ✔ | 2 | | | ✔ | 6 | | | 1 |
| 4* | X | X | X | X | X | X | X | X | |
| 5 | ✔ | 1 | ✔ | 1 | ✔ | 12 | | | 2 |
| 6 | ✔ | 1 | ✔ | 2 | ✔ | 5 | ✔ | ✔ | 2 |
| 7 | | | ✔ | 1 | ✔ | 2 | | | 3 |
| 8* | ✔ | | ✔ | | ✔ | | ✔ | ✔ | 2 |
| 9 | ✔ | 1 | ✔ | 8 | ✔ | 6 | | | 3 |
| 10 | | | ✔ | 2 | | | | | 0 |
| 11 | | | ✔ | 1 | ✔ | 4 | | | 2 |
| 12 | | | | | ✔ | 1 | | | 0 |
| **Total** | 6 | 6 | 8 | 18 | 10 | 41 | 2 | 2 | 20 |

* Nadine's interview did not include Problem 4

**Student:** *Neil*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✓ | 1 | | | ✓ | 3 | | | 3 |
| 3a | | | ✓ | 4 | ✓ | 2 | | | 3 |
| 3b | | | ✓ | 1 | ✓ | 3 | | | 2 |
| 4 | | | ✓ | 3 | ✓ | 1 | ✓ | | 1 |
| 5 | | | ✓ | 4 | ✓ | 3 | | | 2 |
| 6 | ✓ | 3 | ✓ | 2 | ✓ | 3 | ✓ | ✓ | 3 |
| 7 | | | ✓ | 2 | ✓ | 3 | | | 3 |
| 8 | ✓ | 1 | ✓ | 3 | ✓ | 5 | ✓ | | 3 |
| 9 | | | ✓ | 9 | ✓ | 4 | | | 3 |
| 10 | | | ✓ | 1 | ✓ | 1 | | | 2 |
| 11 | | | ✓ | 1 | ✓ | 5 | | | 2 |
| 12 | | | ✓ | 2 | ✓ | 2 | ✓ | | 2 |
| **Total** | 3 | 5 | 11 | 32 | 12 | 35 | 4 | 1 | 29 |

**Student:** *Tanya*

| | Context Category | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Problem** | TL-S (number) | TL-S (count) | TL-W (number) | TL-W (count) | EL (number) | EL (count) | F (number) | O (number) | **Completion Score** |
| 2 | ✓ | 2 | ✓ | 2 | ✓ | 7 | | | 3 |
| 3a | ✓ | 1 | ✓ | 3 | ✓ | 1 | | | 3 |
| 3b | ✓ | 1 | ✓ | 2 | ✓ | 5 | | | 2 |
| 4 | | | ✓ | 4 | ✓ | 4 | ✓ | | 2 |
| 5 | ✓ | 2 | ✓ | 4 | ✓ | 11 | | ✓ | 3 |
| 6 | ✓ | 1 | ✓ | 5 | ✓ | 5 | ✓ | ✓ | 3 |
| 7 | | | ✓ | 1 | ✓ | 1 | | | 3 |
| 8 | ✓ | | ✓ | | ✓ | | ✓ | ✓ | 3 |
| 9 | ✓ | 1 | ✓ | 10 | ✓ | 6 | | | 3 |
| 10 | | | ✓ | 3 | | | | | 3 |
| 11 | | | ✓ | 4 | ✓ | 4 | | ✓ | 2 |
| 12 | | | ✓ | 4 | ✓ | 1 | ✓ | | 2 |
| **Total** | 7 | 8 | 12 | 42 | 11 | 45 | 4 | 4 | 32 |

## APPENDIX T - Four Completed Student Written Responses and Transcripts

### Daniel's Written Responses

3. (a) Evaluate the following:

$$\lim_{x\to\infty} \frac{x^4+4}{x^3-x+5}$$

$$\underline{\infty^4+4} = \frac{+\infty}{+\infty} = \infty$$

$$\frac{(x^2+2)(x^2+2)}{\cancel{x}\cdot x^3-x+5}$$

(b) What does "limit" mean to you?

limit, for myself, represents a barrer or endpoint at which something is possible. For example a swimmer may only be able to swim one mile because that is the "limit" of his or her endurance, similarly in math, though more complex, a limit represents a maximum or minimun possibility.

ie $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12} \rightarrow 0$

4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?

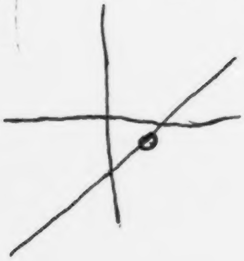
$$y = \frac{2^2 - 5(2) + 6}{2 - 2} = 0 \quad \text{undefined}$$

$$y = \frac{(x-3)\cancel{(x-2)}}{\cancel{x-2}}$$

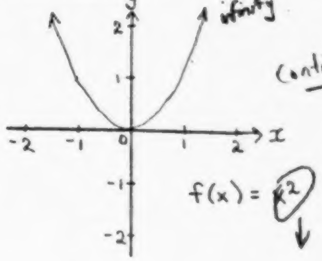
$(2,-1)$

$y = -1$

$y = x - 3$

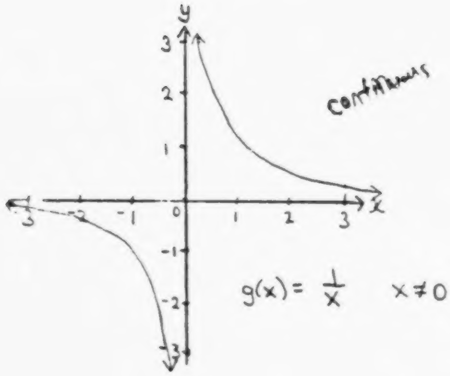


5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.



$f(x) = x^2$

continuous

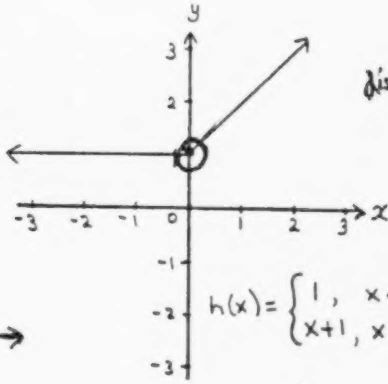


$g(x) = \frac{1}{x} \quad x \neq 0$

continuous

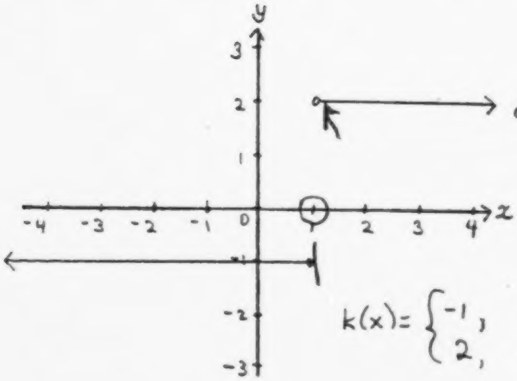
$\nexists$ exist

LHL = RHL



$h(x) = \begin{cases} 1, & x < 0 \\ x+1, & x \geq 0 \end{cases}$

discontinuos



$k(x) = \begin{cases} -1, & x \leq 1 \\ 2, & x > 1 \end{cases}$

discontinuos

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative".

What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?

A derivative is

$$f(x) = 6x^2 + 5x + 1$$

$$\Downarrow \quad \updownarrow \quad \Uparrow$$

$$f'(x) = 12x + 5$$

$$f''(x) = 12$$

$$f(x) = \frac{f(a) - f(b)}{a - b}$$

mathematic definition

7. Find the derivative of each of the following:

$$y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}$$ ⇒ rules regarding sums, quotients

$$3x^2$$

$$F(t) = \left[2t^2 + 3t - 2\right]^{10}\left[3t^{1/4} - 9\right]^7$$

⇒ rules.

$$\left[3t^{1/4} - 9\right]^7$$

inner /outer function

$$\left[\frac{3}{4}t^{-3/4}\right]^7 \qquad \cancel{7}\left[3/4\,t^{-3/4}\right]^{\cancel{7}\,6}$$

8. What interpretation do you have for the following?

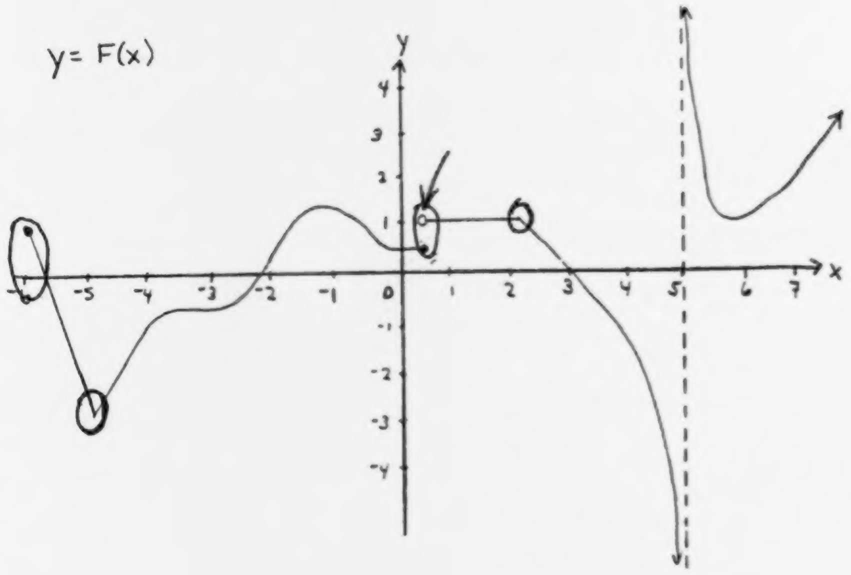
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$0$

⇒

$$\frac{f(x+h) - f(x)}{0} = \text{doesnot exist}$$

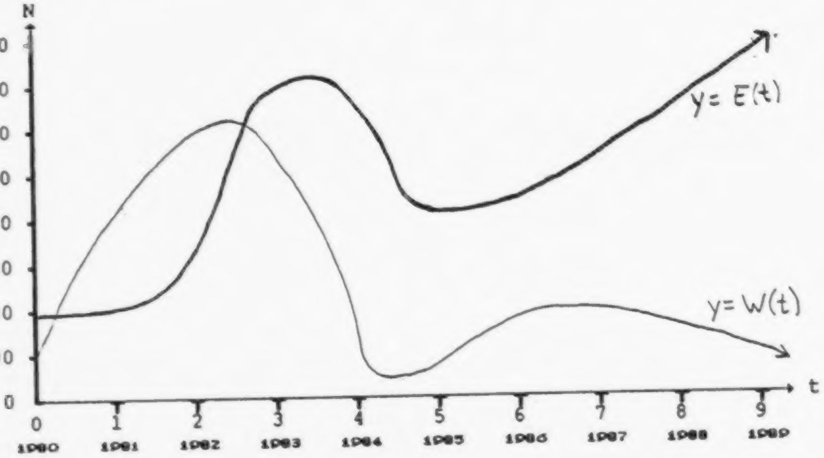
9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?



10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point (0,-2).

$x^2y + y^2 - 3x - 4 = 0$

11. The number of elk in a national wildlife park at the beginning of each year is represented by the function $y = E(t)$ as shown on the graph below. The number of wolves is represented by the function $y = W(t)$, also graphed below.



— (a) At what exact point in time was the number of elk increasing most rapidly?

82 - 83

— (b) During what time period was the rate of change of the number of elk decreasing?

1984 - 85

(c) If you are told that for $0 \leq t \leq 4$ (ie. from 1980 to 1984) the equation for $y = W(t)$ is $W(t) = -100t^2 + 1600t + 500$ (t measured in years), how would you determine all critical points of W?

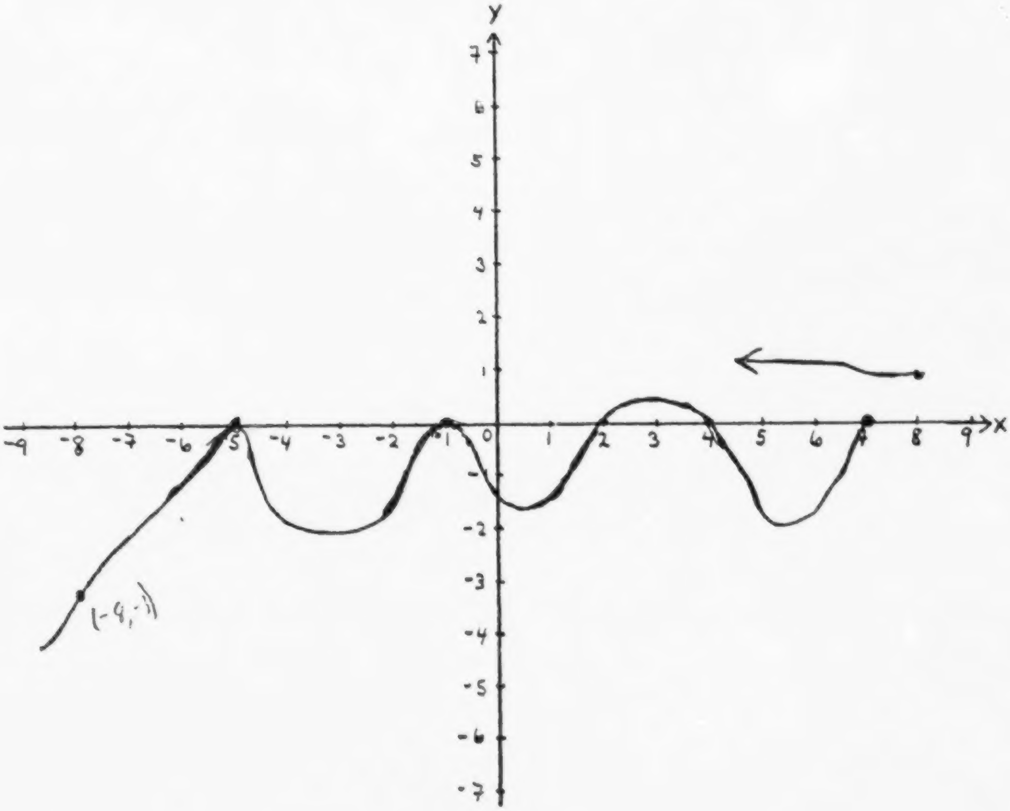
(d) How would you use the critical points found in part (c) to determine the local extrema of W?

— (e) At what point or points in time is the number of wolves not changing?

84-85 , 86 - 87

12. On the axes drawn below, sketch the graph of a function with the following properties:

(a) y coordinate of -3 when x = -8
→ (b) derivative of 2 when x = -5
(c) local maximum when x = -1
→ (d) derivative of 0 when x = 2
(e) slope of 1 when x = 4
(f) when x = 7, a point where the function is continuous but not differentiable
(g) when x>8, f'(x)<0 and f''(x)>0



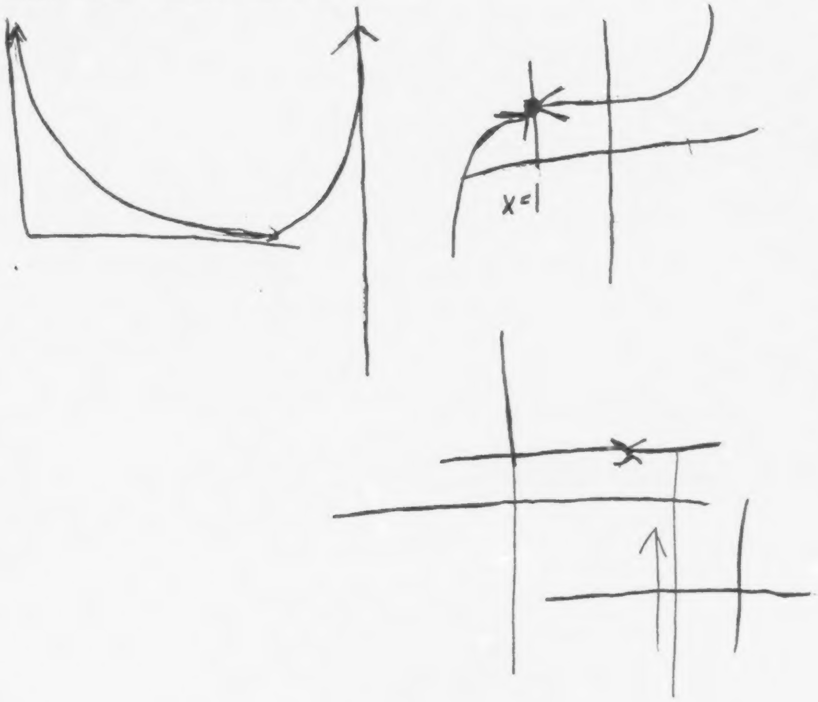
## Sally's Written Responses

3. (a) Evaluate the following:

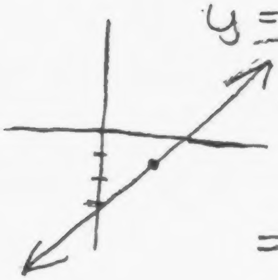
$$\lim_{x \to \infty} \frac{x^{4} + 4}{x^{3} - x + 5} \qquad \infty$$

$$\frac{(\infty)^4 + 4}{(\infty)^3 - \infty + 5}$$

(b) What does "limit" mean to you?



4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?



$$y = \frac{2^2 - 5(2) + 6}{2-2}$$

$$= \frac{4 - 10 + 6}{2-2} = \frac{0}{0}$$

$y = x - 3$

$$\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$$

$y = (x-3)$

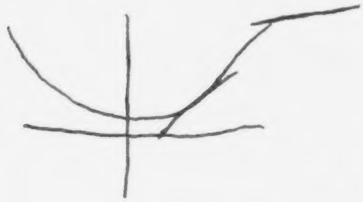
$(2-3)$

$x = 3$

$$\boxed{y = -1}$$

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative".

What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?

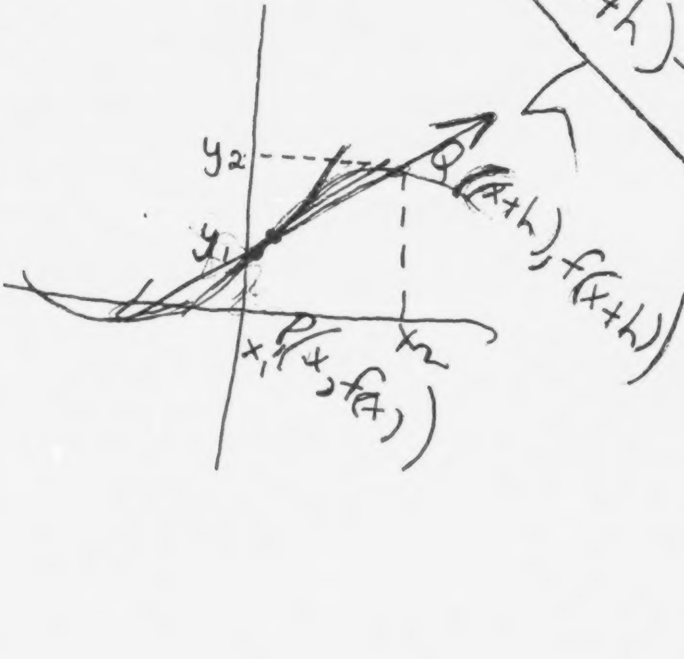


$$f'(x) = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$\lim_{h \to 0} = f(x+h) - f(x)$$



$$\frac{y_2 - y_1}{x_2 - x_1}$$

$\sqrt{x} = x^{\frac{1}{2}}$

7. Find the derivative of each of the following:

$$f(x) = y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}$$

$$f'(x) = \frac{f'\left(x^3 + \frac{1}{x}\right) f\left(\sqrt{x} + 3x^2 + 7\right) - f\left(x^3 + \frac{1}{x}\right) f'\left(\sqrt{x} + 3x^2 + 7\right)}{\left(\sqrt{x} + 3x^2 + 7\right)^2}$$

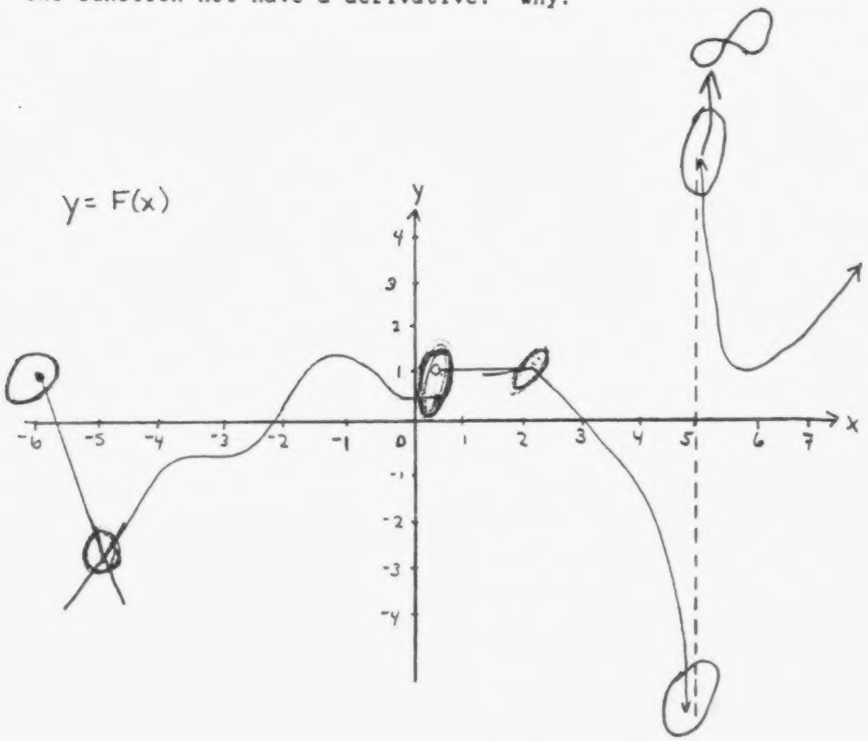
$$= \frac{\left(3x^2 + 1\right)\left(\sqrt{x} + 3x^2 + 7\right) - \left(x^3 + \frac{1}{x}\right)\left(\frac{1}{2}x^{-\frac{1}{2}} + 6x\right)}{\left(\sqrt{x} + 3x^2 + 7\right)^2}$$

$$F(t) = \left[2t^2 + 3t - 2\right]^{10}\left[3t^{1/4} - 9\right]^7$$

$$f'(t) = \left[10\left(2t^2 + 3t - 2\right)^9\left(4t + 3\right)\right]\left(3t^{1/2} - 9\right)^7 +$$

$$\left(2t^2 + 3t - 2\right)^{10}\left[7\left(3t^{1/4} - 9\right)^6\left(\frac{3}{4}t^{-\frac{3}{4}}\right)\right]$$

9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?



10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point (0,-2).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$y^2 = 4$

$-2^2 = 4$

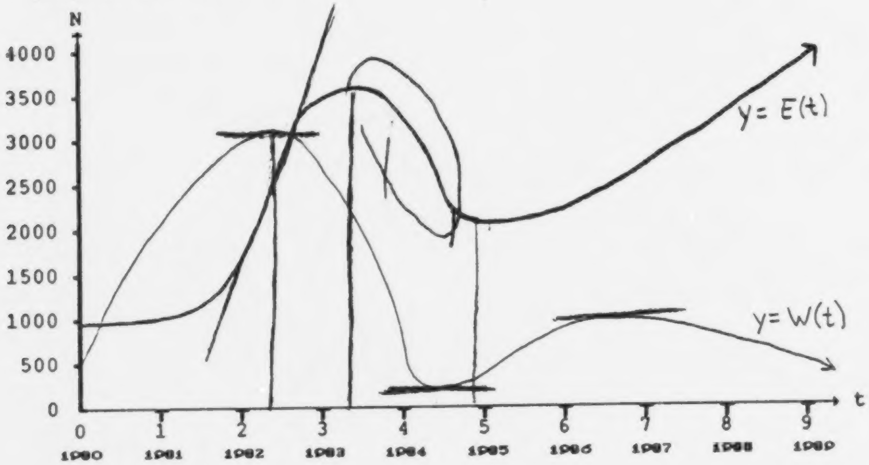
$2x + 2y - 3 \neq 0$

$2(0) + 2(-2) - 3 \neq 0$

$0 + -4 - 3 \neq 0$

$-7 \neq 0$

11. The number of elk in a national wildlife park at the beginning of each year is represented by the function $y = E(t)$ as shown on the graph below. The number of wolves is represented by the function $y = W(t)$, also graphed below.



(a) At what exact point in time was the number of elk increasing most rapidly?

[illegible] 1982.

(b) During what time period was the rate of change of the number of elk decreasing?

1983 -1984

~~(c)~~ If you are told that for $0 \le t \le 4$ (ie. from 1980 to 1984) the equation for $y = W(t)$ is $W(t) = -100t^3 + 1600t + 500$ (t measured in years), how would you determine all critical points of W?

~~(d)~~ How would you use the critical points found in part (c) to determine the local and global extrema of W?

(e) At what point or points in time is the number of wolves not changing?

Apr. 1982 June 1984 Aug. 1986

12. On the axes drawn below, sketch the graph of a function with the following properties:

(a) y coordinate of -3 when x = -8

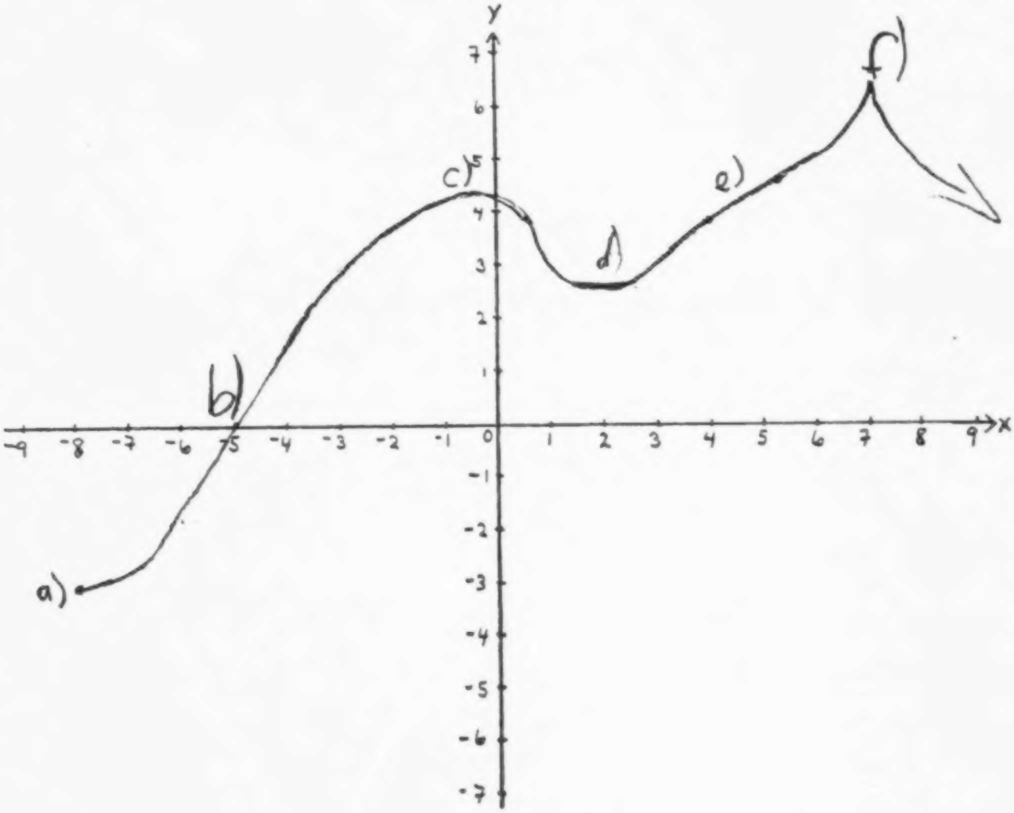
(b) derivative of 2 when x = -5

(c) local maximum when x = -1

(d) derivative of 0 when x = 2

(e) slope of 1 when x = 4

(f) when x = 7, a point where the function is continuous but not differentiable



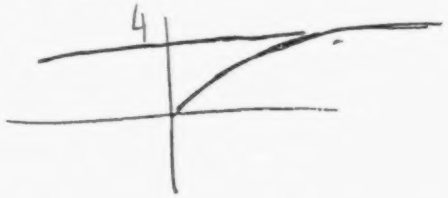
## Jennifer's Written Responses

2. For each of the following sequences of numbers, decide whether the sequence has a limit.

   If so, what is this number?

   $1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots$

   $3.9, \quad 3.99, \quad 3.999, \quad 3.9999, \quad 3.99999, \quad 3.999999, \ldots$



3. (a) Evaluate the following:

$$\lim_{x \to \infty} \frac{x^4 + 4}{x^3 - x + 5} \qquad 1$$

$$\frac{\frac{x^4}{x^4} + \frac{4}{x^4}}{\frac{x^3}{x^3} - \frac{x}{x^4} + \frac{5}{x^4}}$$

$$\frac{1+0}{\cancel{\frac{1}{x}} \quad \frac{1}{x^2}}$$

$$0 + 0$$

(b) What does "limit" mean to you?

$$\lim_{x \to 1} f(x) = \boxed{2} \rightarrow$$

4. What can you say about the function $\boxed{y = \frac{x^2 - 5x + 6}{x - 2}}$ at x = 2?

$$y = \frac{4 - 10 + 6}{0}$$

$$-1 = x^2$$

$$\frac{(x-3)(x-\cancel{2})}{(x-\cancel{2})}$$

$$y = -1 = \boxed{\frac{x^2 - 5x + 6}{x - 2}}$$

$$-x + 2 = x^2 - 5x + 6$$

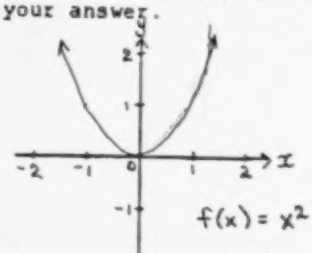
~~2~~

$$x^2 - 4x + 4$$

$$(x-2)(x-2) = 0$$

$$x = 2$$

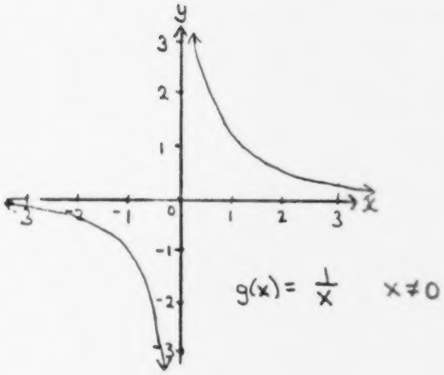
5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.

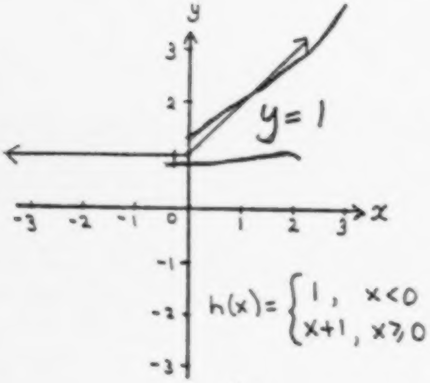


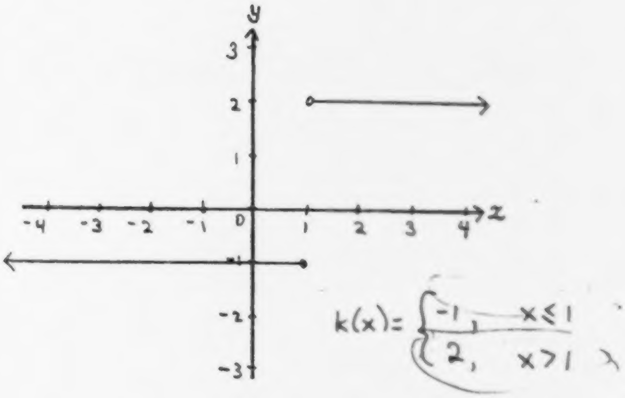
$$\lim_{x \to \infty} f(x) = x^2$$

$$= \frac{x^2}{x^2}$$

$$= 1$$







$k(x) = -1$

$k(x) = 2$

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative".

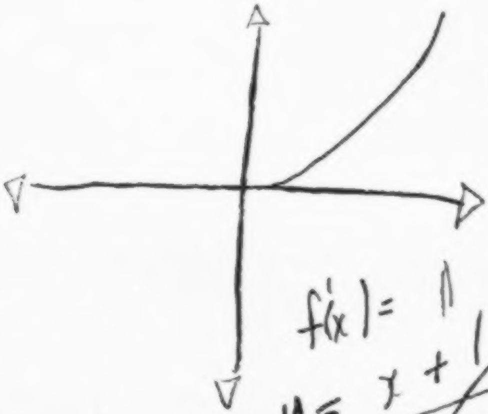
   What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?

$f(x) = x^2 + x^3 + 5$ ✓

$f'(x) = 2x + 3x^2$

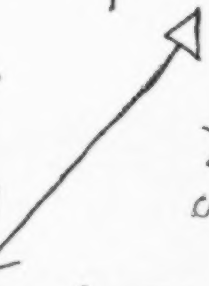
$y =$ ~~$x^2 + x^3$~~

y-intercept $= 5$



ro $q^3$

4



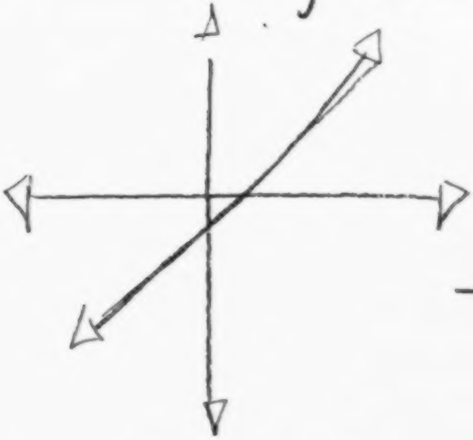
$x(2+3x)$

critical points

$x = 0$ $x = \frac{-2}{3}$

$f'(x) = 1$

$y = x + 1$



| | $x$ | $(2+3x)$ |
|---|---|---|
| ~~$(-\infty, 0)$~~ | | |
| (0 $(-\infty, -\frac{2}{3})$ | $-$ | $-$ |
| $(\frac{-2}{3}, 0)$ | $-$ | $-$ |
| $(0, +\infty)$ | $+$ | $+$ |

$f'$ +

+

+

7. Find the derivative of each of the following:

$$y = \frac{\overset{u}{\left( x^3 + \frac{1}{x} \right)}}{\underset{v}{\sqrt{x} + 3x^2 + 7}}$$

$$y' = \frac{\left(x^3+\frac{1}{x}\right)\left(\frac{1}{2}x^{-1/2}+6x\right) - \left(3x^2-\frac{1}{x^2}\right)\left(\sqrt{x}+3x^2+7\right)}{\left(\sqrt{x}+3x^2+7\right)^2}$$

$$F(t) = \left[2t^2 + 3t - 2\right]^{10}\left[3t^{1/4} - 9\right]^7$$

$$F'(t) = 10(2t^2+3t-2)^9(4t+3)(3t^{1/4}-9)^7$$

$$+ \left[2t^2+3t-2\right)^{10} 7(3t^{1/4}-9)^6\left(\frac{3}{4}t^{-3/4}\right)$$

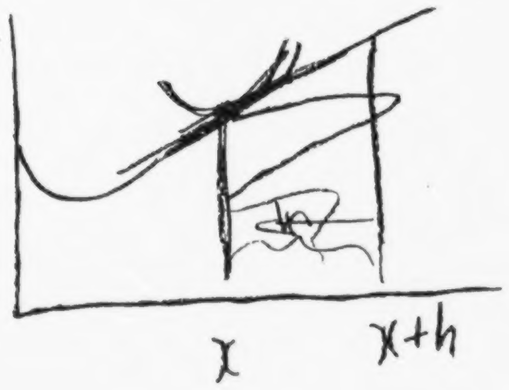
$$\underline{=}^{10}(40t+30)(2t^2+3t-2)(3t^{1/4}-9)^7$$

$$8t^3+12t^2-8t+6t^2+9t-6$$

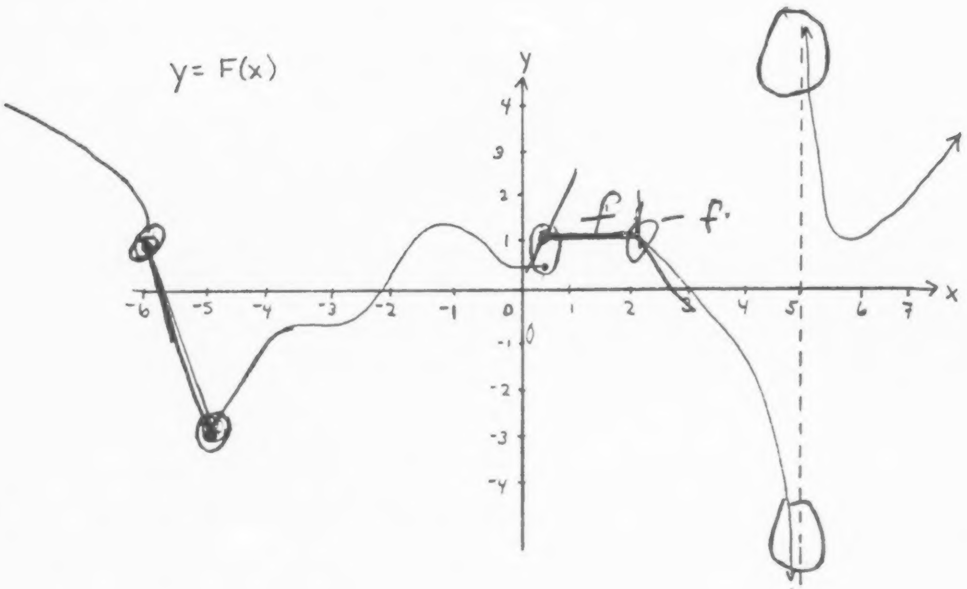
$$10\ (8t^3+10t^2+9t-6)(3t^{-1/4}-9)^7$$

8. What interpretation do you have for the following?

$$\lim_{h \to o} \frac{f(x+h) - f(x)}{h}$$



9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?



$y = |x|$

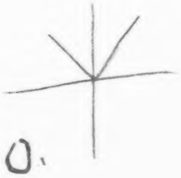
$x \leq 0.$

$x > 0$

$y' = |$[illegible]

$y = |1|$

$y' = 1$



10. Find the slope of the tangent line to the curve $x^2y + y^2 - 3x = 4$ at the point $(0,-2)$.

$$2xy + x^2y' + 2yy' - 3 = 0$$

$$x^2y' + 2yy' = 3 - 2xy$$

$$y'(x^2 + 2y) = 3 - 2xy$$

$$y' = \frac{3 - 2xy}{x^2 + 2y}$$

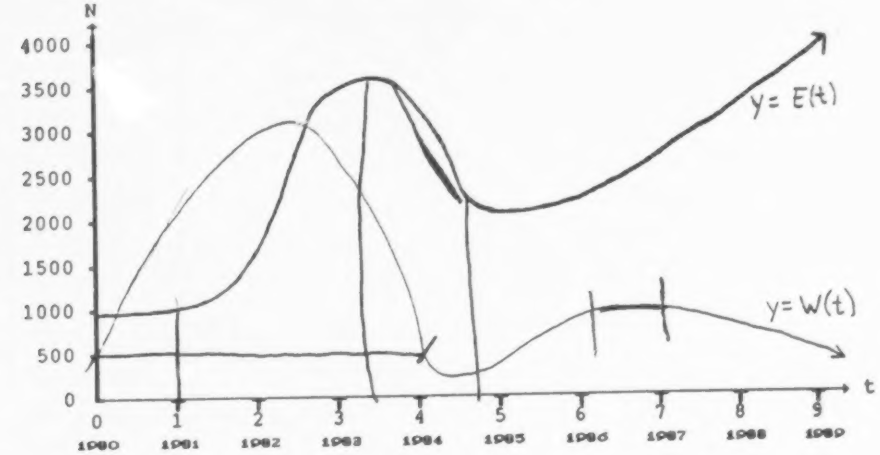
$$y' = \frac{3 - 2(0)(-2)}{(0)^2 + 2(-2)}$$

$$= \frac{3 - 0}{-4}$$

$$y' = \frac{-3}{4}$$

✓

11. The number of elk in a national wildlife park at the beginning of each year is represented by the function $y = E(t)$ as shown on the graph below. The number of wolves is represented by the function $y = W(t)$, also graphed below.



(a) At what exact point in time was the number of elk increasing most rapidly?

81 - 83

(b) During what time period was the rate of change of the number of elk decreasing?

83 - 85

(c) If you are told that for $0 \le t \le 4$ (ie. from 1980 to 1984) the equation for $y = W(t)$ is $W(t) = -100t^3 + 1600t + 500$ ($t$ measured in years), how would you determine all critical points of $W$?

(d) How would you use the critical points found in part (c) to determine the local extrema of $W$?

(e) At what point or points in time is the number of wolves not changing?

86 - 87

12. On the axes drawn below, sketch the graph of a function with the following properties:

(a) y coordinate of -3 when x = -8 (-8, -3)

(b) derivative of 2 when x = -5 $f'(5) = 2$ @ $x = -5$

(c) local maximum when x = -1

(d) derivative of 0 when x = 2

(e) slope of 1 when x = 4

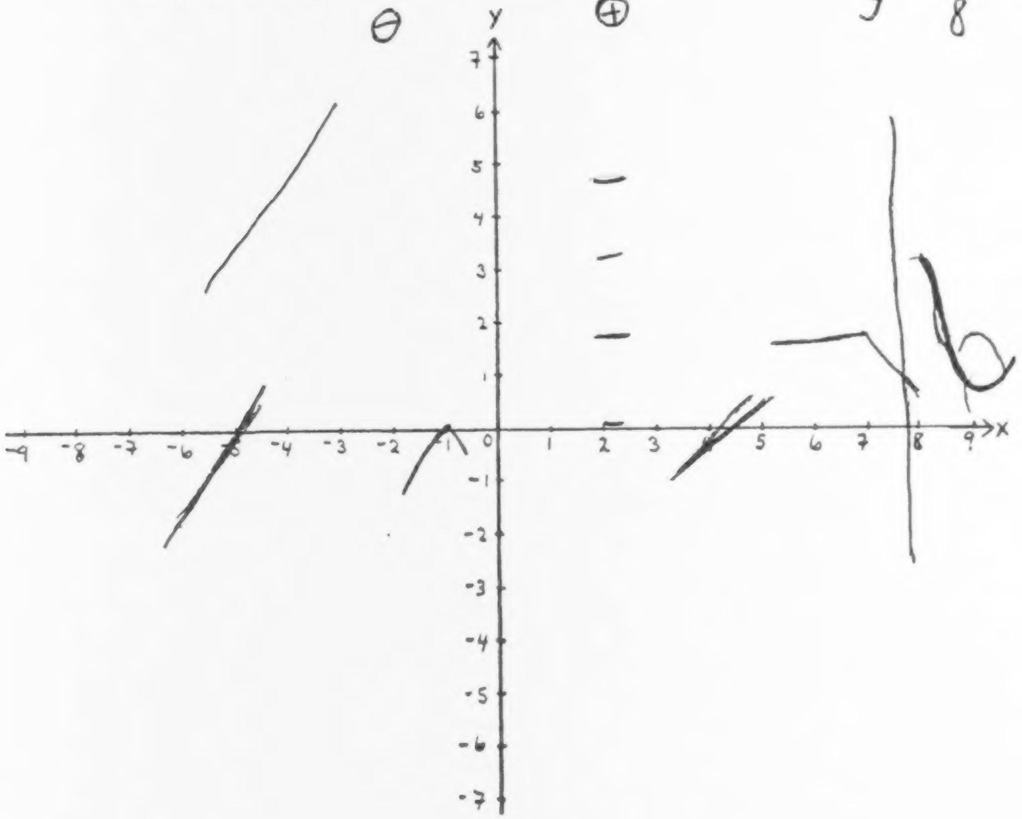
(f) when x = 7, a point where the function is continuous but not differentiable

(g) when $x > 8$, $f'(x) < 0$ and $f'(x) > 0$.

S - 8

y 3

y $\frac{-15}{8}$



## Tanya's Written Responses

2. For each of the following sequences of numbers, decide whether the sequence rounds off to a particular number.

If so, what is this number?

$1,\ \frac{1}{10},\ \frac{1}{100},\ \frac{1}{1000},\ \frac{1}{10000},\ \frac{1}{100000},\ \ldots$ 0

3.9, 3.99, 3.999, 3.9999, 3.99999, 3.999999, . . .

$$\left.\begin{matrix} 4-\iota_0 \\ 4-dx \end{matrix}\right\}$$ $\iota_0$ + $dx$ are any positive infinitesimal

4

3. (a) Round off the following:

$$\frac{M^4+4}{M^3-M+5} \quad \frac{\frac{M^4}{M^4}+\frac{4}{M^4}}{\frac{M^3}{M^4}-\frac{M}{M^4}+\frac{5}{M^4}} = \frac{1+\frac{4}{M^4}}{\frac{1}{M}-\frac{1}{M^3}+\frac{5}{M^4}} = \frac{1+dx}{dx}$$

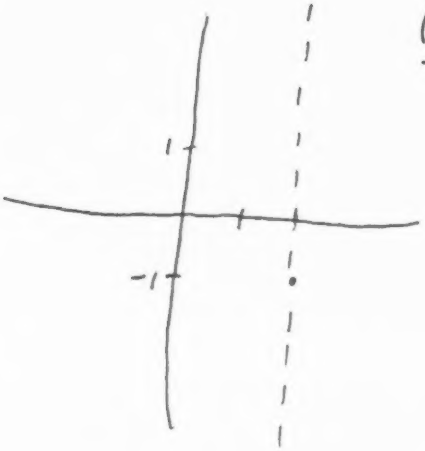
$$\rightsquigarrow \infty$$

(b) What does "round off" mean to you?

the answer is the number closest to the question

$$\frac{5+3}{2} \quad \frac{8}{2} = 4$$

4. What can you say about the function $y = \frac{x^2 - 5x + 6}{x - 2}$ at $x = 2$?

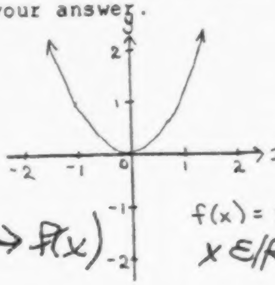


$$\frac{(x-3)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$\boxed{-1}$$

$y = x - 3$

5. For each function given below, determine if it is continuous or discontinuous. Give reasons for your answer.



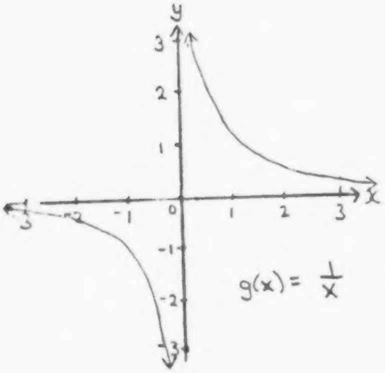
$f(x) = x^2$

$x \in \mathbb{R}$

$- f(x+dx) \to f(x)$

$y = x^2$

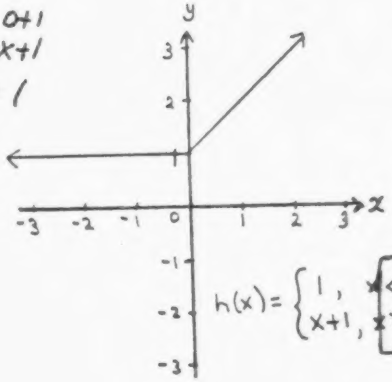
$y(x) \to (x+dx)^2$



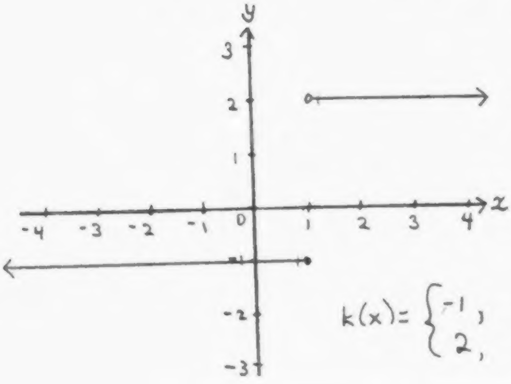
$g(x) = \frac{1}{x} \quad x \neq 0$

$h(0+dx) \to \overset{0+1}{x+1}$

$h(0-dx) \to 1$



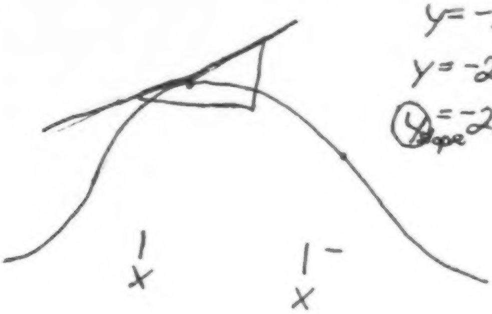
$h(x) = \begin{cases} 1, & x<0 \\ x+1, & x \geq 0 \end{cases}$



$k(x) = \begin{cases} -1, & x \leq 1 \\ 2, & x > 1 \end{cases}$

6. A friend of yours who recently completed high school mathematics is wondering what calculus is all about because he/she has heard you frequently use the word "derivative".

   What short explanations, sentences, or examples would you use to explain to your friend what the "derivative" is all about?



$y=-x^2$

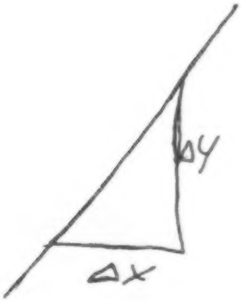
$y=-2x'$

$\enclose{circle}{y}_{\text{slope}}=-2x$

$f(x)=y$

$y+dx-y$

$$\frac{f(x+dx)-f(x)}{dx}$$



7. Find the derivative of each of the following:

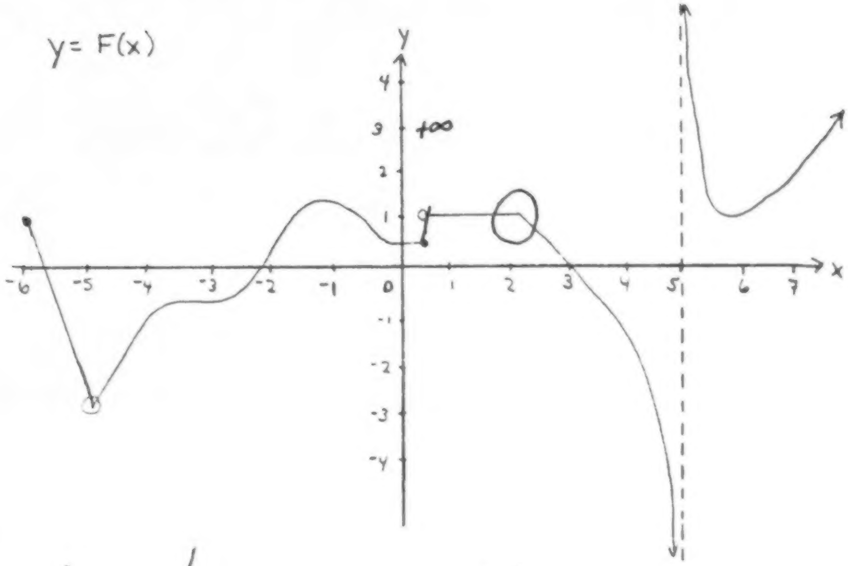
$$y = \frac{x^3 + \frac{1}{x}}{\sqrt{x} + 3x^2 + 7}$$

$$F(t) = \left[2t^2 + 3t - 2\right]^{10}\left[3t^{1/4} - 9\right]^7$$

$$F'(t) = 10(2t^2+3t-2)^9 \cdot (4t+3)\left[(3t^{\frac{1}{5}}-9)\right]^7 +$$

$$7(3t^{\frac{1}{4}}-9)^6 \cdot \left(\frac{3}{4}t^{-\frac{3}{4}}\right)\left[2t^2+3t-2\right]^{10}.$$

9. The graph of $y = F(x)$ is given below. At which points does the function not have a derivative? Why?



$x+dx \rightsquigarrow y(x)$

10. Find the slope of the tangent line to the curve
$x^2y + y^2 - 3x = 4$ at the point (0,-2).

$2xy \cdot y'(x^2) + 2y \cdot y' - 3 = 0$

$2xy \cdot y'x^2 + 2yy' = 3$

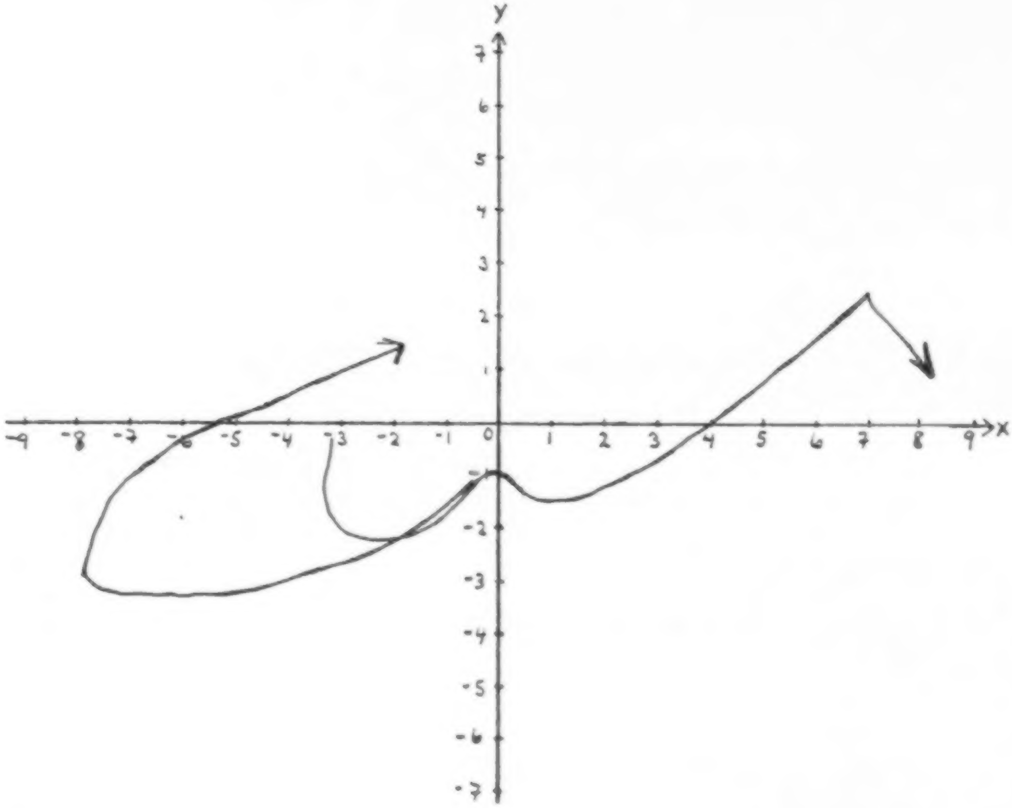
$y'(2xyx^2 + 2y) = 3$

$$y' = \frac{3}{2xyx^2 + 2y}$$

$$y' = \frac{3}{-4}$$

12. On the axes drawn below, sketch the graph of a function with the following properties:

    (a) y coordinate of -3 when x = -8
    (b) derivative of 2 when x = -5
    (c) local maximum when x = -1
    (d) derivative of 0 when x = 2
    (e) slope of 1 when x = 4
    (f) when x = 7, a point where the function is continuous but not differentiable



**Daniel's Transcripts**

Interview 1

*Problem 1*

- What would you say to this person? Calculus is um the study of functions and how they can be broken down and applied and graphed, and that's about it I'd say. Broken down and applied. Derivatives, intercepts, graphing, um (pause).
- Those are some of the parts of calculus?
- Yeah.
- Do you have any way of summarizing in 10 words or less, calculus is about this, such and such?
- No.

*Problem 2*

- (pause) Well. (pause) I don't believe this one does have a limit. Infinite.
- Why do you think that?
- Um. Because the progression here. Three point nine, three point nine nine, all the way through is um is not it's not narrowing to a certain number. It's what do you call it? It's getting larger. And this one um is getting smaller so I think it's approaching zero.
- Can you say more about that? This first one.
- Um. The first number in the first one is one, and then the second one is one tenth. One one hundredth. These numbers are getting progressively smaller. And while the numerator is constant the denominator gets larger which leads to a smaller number. Which would seem to me it means it would eventually somewhere way way down it would approach to zero.
- Would it ever actually reach zero?
- No. Because it would never, well the way I see it probably would never ever equal zero. That would make it an infinite number. I'd say it approaches zero, but it would never actually reach zero.
- Okay. And can you say any more about the second one? Or even repeat what you said to me.
- Well, um this number is obviously approaching a number close to four. Um. But I don't think it will reach four because the question is three point nine and each number after that has an additional nine after it right. And that sequence just continues forever. It doesn't really make the number, it makes the number larger by a fraction which would eventually lead to a, coming close to four.
- Then, but previously you said it was infinite. Now you're saying it comes close to four.
- Yeah. Well I believe this one, the second one is infinite. Like I mean it's because it's three point nine it's close to four. Maybe I expressed it wrong when I said it comes to four. I believe it's an infinite expression because it goes, just by looking at the numbers here. Three point nine nine. I can't base it on any theorem because I don't know, but just by looking at the sequence of numbers I can't see how that would have a limit because all it is is an addition of a number.
- So am I right in saying it's an infinite expression, but you see it as getting near four?
- Well in saying that I'm saying it has a limit, right, if I say that it gets near four. See. (pause). It does get nearer to four but I don't , I guess I'm hazy on what the definition, what the difference is between this. This one I'd say definitely approaches zero. No wait. I don't know. Now I'm all confused. But anyways, um, I guess it's either infinite or it approaches four. It's not both right.
- But do you know which? If you're unsure you can say so.
- Okay. Well then I'm unsure because it looks to me like it is approaching, well it obviously is getting closer to four. Because three point nine nine is smaller than three

point nine nine nine, right. So it's obviously getting larger and as it gets larger it goes towards four. However, um, that sequence doesn't necessarily stop there so I don't think it has, that's how come I don't think it has a limit.

- Okay.
- It's still getting bigger even if it did, well, no. It's going to be three point nine nine forever though. Because even though that's getting closer to four why would that ever go up to four. It goes on forever.

*Problem 3a*

- Okay. x approaches infinity. (pause) (writing) (pause) (erasing heard)
- Can I ask you what you've been doing?
- Just, there.
- What is it you're trying to do?
- Ah, I was just going to evaluate the following limit. I evaluate the limit by first of all getting it in terms I can understand and evaluating it as x approaches infinity.
- So what is it you're doing here? Trying to put it in terms you understand?
- Yeah. By reducing these. (pause) Let's see. (erasing) I know there's a formula here, well not a formula but a rule to evaluating limits here that we ah, just, quotient. (pause) Well, I'll try this. Because x here um what I want to do is I'm just going to um, I don't know if that's right.
- Can you tell me what you were going to say? Whether it's right or not.
- Okay. Ah, as I see it there's two possible ways. There's one or another way to do it, and I'm not sure which is right. I have a feeling that the way I am about to show you is wrong, but what I was going to do is say as x goes to infinity, and you can represent that by essentially something like that.
- Okay. Infinity to the fourth.
- Plus 4. Which just equals a large infinite number. And here, infinity to the third minus an infinite number is still a large infinite number. So you have an expression that is somewhat undefined or is it is large infinite. But that's not the way I think this problem is done. That's just the only way that is coming to mind. The other way this problem could be done is, well not the other way, the correct way, has something to do with the fact that this is a quotient. And wait a second. Okay. (pause) I'm just thinking, Um, (pause) From the left or from the right. (pause).
- Can you tell me what you're thinking? You're thinking plus and minus infinity, I know that.
- Yeah, I'm just, I know that's there's, I know that there's another way that's the correct way to do the problem. I know there's a correct way to do it, but I'm not remembering it.

*Problem 3b*

- Alright. What does limit mean to me? (pause) (writing for along time)
- Okay. Can you tell me what you've put there?
- This is what I have written here. The limit for myself represents a barrier or end point at which something is possible. For example, a swimmer would only be able to swim one mile because that is the "limit" of his or her endurance. Similarly in math, though more complex, a limit represents a maximum or minimum possibility. Ah, for example one half, one quarter, one eighth, one twelfth. This progression is near the limit of zero.
- And could it ever reach zero?
- And once again I draw reference to the first question I did in which the same problem boggled my mind. So as you can see I think I have an idea of what the English definition of limit means, but in mathematical terms I think it's probably pretty hazy.
- Alright. Let me just ask you, you say it's a barrier and an end point.

- Well maybe that wasn't quite correct. I was thinking like ah, like if you have a progression of numbers it can only reach this number. That's as high as it can reach, or as low as it can reach. A maximum or minimum.
- Can it go beyond that and come back?
- Um. I suppose in certain instances it could. Um. I said here it represents a barrier or end point, but ah I guess if you're thinking in terms of something rising. Yeah, it could go beyond that and come back. I would have no idea about how to go about doing that.
- So it could go beyond it and come back. And could it ever actually get exactly to that limit? This one you're not sure about, but is there cases where it would maybe definitely reach the limit?
- Yes. There are definitely cases. I don't have any off the top of my head straight off, but I think there are definitely some cases where it could reach it's limit. Like, in fact I could almost be sure of that because I remember it from class. But maybe I'm wrong.

*Problem 4*

- Alrighty. (pause) (writing) Um. Doesn't exist.
- Why?
- Because at x equals 2 the number in the denom, sorry the bottom, the denominator, whatever, equals zero. Which ah cannot occur for a fraction. 'Cause that makes it undefined. Undefined. Oh, hang on. Or let me take this a step farther. Um. (long pause) Okay. Um. Yeah. (pause) Yeah. (pause) (mumbling) (writing)
- Can I ask what you've done at this point?
- Okay. Well, the first thing I did was I just straight across substituted in at x equals 2 and I found that the denominator went to zero. And then I ah took it here and I, what do you call it, I broke it down, I didn't , I described this function in terms of two ah, I took it down to its factors that's what I did. So I had x minus 3 times x minus 2, all over x minus 2. And ah, so then I substituted 2 in and I got, I took out as an ordered pair, so if y equals, if x equals 2, y will equal negative one. And now um I'm just looking for a way to see if I can maybe get a slope or something from that so I can graph it. Um. (pause) Or actually, (mumbling). (pause)
- What are you thinking?
- Um. I just, I'm just not sure how to go about from this stage now to graph it. I'm not sure what would be the best, so.
- Are you trying to graph it, is that right?
- Yup.
- And what is it you're trying to graph? What expression?
- Ah, just the function at x equals 2. See one thing I don't understand is how come at the beginning when I substituted everything in it came to zero. Because that shouldn't come to zero. It doesn't matter if it's been reduced or not. Which I think means that the graph doesn't exist at this point. Like there's a hole in the graph.
- What would it look like then, maybe?
- Um. (pause) Well actually I think it might be a straight line, but I'm not sure. Um. Say something like this. (writing) I've got to erase this one point. There'd be a hole right there, or something like that. I don't know. Something along that line.

*Problem 5*

- Continuous or discontinuous. Okay. (pause) Continuous. Continuous because there's no break in the graph.
- Alright. Can you say anything else?
- Um. Ah, well there's no break in the graph and it appears to go to infinity like this forever. And that is further demonstrated by the function x squared because what do you call it, exponential growth? Like it will cause this thing to go on forever, and also from looking at the graph I see it's smooth. There's no breaks or spaces where the

graph doesn't exist. So that's why I'd say it's continuous. Um. (pause) (mumbling) x not equal to zero.

- (pause) I'll just come back to that one.
- Okay.
- Hm. (pause) Okay, this one is discontinuous. (pause) I'll just write down what I think and then I'll come back.
- Okay. Do it in any order you want.
- Actually, (long pause) I find this one tricky here. The one with the two hyperbolas. Because ah um, on one hand I think the graph is smooth, both the hyperbolas are smooth and approaching infinity. But there is a, but x can't equal zero, which seems to me to provide a discontinuity. But I don't think that provides a very strong argument so I'll say that is continuous. Because I'm going to go with my usual hunch because these seem continuous and therefore.
- For this third one you wrote discontinuous and erased it. Then rewrote it again. Can you tell me what thought process went on?
- Alright, Um. I remember reading in my calculus textbook something about continuous functions are smooth ah, with no breaks. And I was trying to remember if it also included what do you call it, ah, sharp turns in a graph. Because I believe that makes it discontinuous at this part right here. Because it's stopping and taking another direction. Um.
- Is that what you mean by smooth?
- Pardon?
- Well, what do you mean by smooth?
- Um. Well. See this, a continuous graph, or um that constitutes a smooth. To me what really is a blatant discontinuity is a hole, a cusp. Which I believe a cusp is. Which that's how come I was thinking this. See I'm not sure about this one. Whether it's discontinuous or continuous because it's not, it seems to me I don't see why this wouldn't be continuous because it's just going off like one, like it's just taking one direction and going off in another. It's not a hole in the graph. But I believe that when I look at it maybe the whole fact that it is taking another completely new direction shows the end of one function, or the end of one set of requirements and the beginning of another. Without flowing together. Like there's a disjoint. Um. For this one I would call it discontinuous because there's a hole in the graph at one here. And here ah, at negative one, this is both on the x-axis. It exists at one, like that's where it starts. Like it doesn't look continuous. I suppose down here it does become continuous because it ah it starts and goes forever. There's no place on this it doesn't exist. But here it does not exist at the coordinates one, comma two. There's a hole in the graph and therefore it's a discontinuous function.
- Okay. But this part is continuous. So what would you say for the whole graph?
- Okay. Yeah, what am I saying? Another thought just occurred to me too. That if it's discontinuous here at one it's discontinuous on the, like I just said right here that it's discontinuous here because there's a hole there. And the one one right beneath it, at the same coordinate on the x-axis, I said was continuous. Well that's correct because there's a hole here and also this one just starts to exist here. It doesn't, oh there's one other thing. Now I remember. It just came to me. Left-hand has to equal right-hand. Um. I'm thinking of limits but still I know in some respect that this theory is transferred onto continuity. Because when you're looking at a function to graph it you ah, see that's how come this one is discontinuous because the left side doesn't equal the right side. Now not that this does. Now let's see. That's a continuous function. It's something like that. I doesn't equal. Oh I know, it has to exist on the left-hand side and right-hand side for it to be continuous. See here it doesn't exist on the left-hand side and here it doesn't exist on the right-hand side. That's my logic anyway.

*Problem 6*

- (pause) Hm. (pause) (writing) (long wait) I've never had to define derivative before. Well, I have in certain terms. Like we've defined derivative as ah f of a minus f of b all over a minus b. That's one of our definitions of derivative, mathematically.
- Can you say more about this?
- About this definition?
- Yeah, the f of a minus f of b all over a minus b.
- Not really. I just have it memorized and I use it. I don't particularly understand it. Um. Ah. No. I don't understand it well enough. I just know when to use it. Say, using a definition of derivative. That's what they refer to there. However, a friend of mine who wants to know about high school calculus, I'd tell him not to take it. I took it because I thought it would be easier than French. Um. A derivative is basically, I'm going to try to explain this rather than write it. A derivative is basically to me um a simpler expression than this one. What I've got written down there. 6 x squared plus 5 x plus 1. Um. Like um. See I know the rules for deriving a derivative, and there's a variety of them depending on whether you're dealing with like, depending on what you're dealing with. I'd say that a derivative may be along the lines of a root of something like this. And you can take a first derivative test which is the first derivative. And a second derivative which is the derivative of the first derivative.
- Can you say more about this notion of a root?
- Okay. Um. Um. I don't know if a root is the right word because probably mathematically speaking that's probably wrong.
- But it means something to you, right?
- Like a derivative to me is like smaller, it's like somehow made into this. Like it's like a factor of this but it is not this whole expression. Um. 'Cause you can't look at that and say well if I square that back up I get this back again. It doesn't work that way because the process for coming to here is not anything that you'd associate with the word root. But it is to me like ah ah I guess a root is the word I'd describe to this, other than derivative. I mean derivative is a perfect word for it because that's what I consider it as. This is a derivative of this. I don't really know the words to explain it but um the notion of it is I guess something like. Okay, if you take ah say pencils and you have a whole pencil and you take off um you take off, you cut the pencil in half. You still have the whole pencil. You can still use it and you can still um get information, or use the pencil to write and things like that. And ah, it's smaller so maybe it's more useful too. You can carry it in your pocket or something. And that's how I look at derivative. It's kind of like another way of expressing this function um to find important information.
- Can you think of any examples of where derivatives are used? In explaining to your friend what they're about, sort of a real world thing that they're about?
- Ah.
- Or a picture even of some kind?
- Yeah. Okay. Okay. Just wait a second here. Um. (pause) Okay. We did one example with distance travelled. Well I have no mathematical examples off the top of my head, just they as they always tell you in math, although it has applications in later life, um, which is believable I suppose. And they give us problems that could be applied to everyday life. And there's lots of those. Like derivatives can be applied to problem solving in math, um.
- Can you think of anything right now?
- Um. (pause) Well, I know that later on in economics, like part of the reason math, calculus is an important course is because of the concept of derivative. And I know that in economics some application of derivative is necessary. So maybe there's some examples that could be used in investment, or banking or something. I don't know. Mortgages or loans, the derivative is used to help calculate.

*Problem 7*

- I don't remember all this. I just kind of forgot once I studied it. Oh well. Let's see. Oh. Well. That isn't going to help me out. I was going to say because there's a root here it can't be that. But I was thinking of a totally different. I was thinking about limits or something. Can you take a derivative of a root? (pause) One thing I was going to say about using these. I realize um for ah solving derivatives there's rules that you use if you're doing sums, and if you're doing divisions, and if you're doing polynomials. And if you're doing like multiplication of derivatives. I understand there's rules.
- Do you remember the rules?
- They're very very hazy in my mind. Um. But. (unclear) Okay if I say there's rules.
- Okay, but do what you can with it.
- Um. (pause)
- If it's too much because you don't remember all the rules, could you do for instance what's inside this second bracket? Forget the rest of it there. Just do what's inside the bracket.
- Yeah. Because I know there's something to do with the fact that there's an inner and outer function. You want me to write that?
- Sure.
- (writing) Um. Let's see. I think. We'll do inner first. What's three times one quarter? (pause) (writing) One minus one quarter. (pause) (writing) (long pause) (unclear for a few words)
- Alright. Could you do just the top part of this one?
- Um. (pause)
- What's confusing you?
- I'm just trying to remember what the rule is for that. Um. That's equivalent of one to a certain extent. Um.
- It's one over x that's the problem right?
- I don't believe it exists. I think the derivative of that is zero so maybe this would be the answer for the top part.

*Problem 8*

- Um. (pause) What interpretation do I have for the following? Then you give me a (unclear) that supposedly does not exist. Okay. Well. Since h is zero and this is all over h that means that it's, oh my god. Over zero. Which does not exist. The reason that does not exist is because a problem can't be divided by zero.
- Do you know why that is?
- Can't be divided by zero?
- Yeah.
- Um.
- It's a tough question.
- Well, I have no mathematical explanation. Do you want me to try to say it in words?
- If you have any ideas.
- Um. (pause)
- It is a tough question. I'm curious if you had any ideas.
- Okay. Well, I believe that you can't divide something by something that doesn't exist. Something like, it's like me saying that ah, I'm going to buy, I'm going to buy some apples at the store but I have no money. Sort of in a sense I can't take this function and divide it by something that isn't there. And yet that's exactly what this question calls for.
- Okay. So you do have any interpretation of it? You've said that it doesn't exist.
- The reason it doesn't exist is because you can't take something that is nonexistent or a quantity that doesn't exist and divide it by a quantity that does exist. And say get a number out of that. Get a quantity of that. Because essentially it's not going to be there.

*Problem 9*

- Um. At what point does the function not have a derivative? As I said earlier the derivative I thought was a smaller form of a function. Um, like a root of a function but not exactly in the classical terms of the word. And ah I'm just trying to figure it out here on the graph where a particular function of the graph would not have any smaller. I suppose it wouldn't have a derivative if it was at a maximum or a minimum. Maybe that's just too (unclear).
- Why is that?
- Um. Well because as I said though the derivative is a smaller form of the function, but if it's at an absolute minimum it's as constant as it gets. See like here, right here, negative 6, I do not believe that has a derivative at that point because this is a constant. It's a ah ...
- What's a constant?
- A number with no variable. I guess all these are constant. (pause) At which points does the function not have a derivative? (pause) Maybe. (pause) Well, I'll tell you what I'm thinking. Um. I'm just trying to decide where they would not have a derivative and I'm looking at possibilities. It could either be where there's a hole, where it's solid, not having a derivative. And I was thinking maybe that's the case. But I don't think it has a derivative at the hole because at that point it does not exist. But as for these places where the graph starts, I don't see why a derivative wouldn't exist. Then the other place I was looking at was ah where they go infinity. So a derivative at infinity here at that point on the graph. Even as the graph approaches here, is there a derivative? At you know 5 or whatever. Or something. And ah, I think I'm opening a fresh can of worms. I wasn't sure how to describe that. That is beyond my realm of understanding right now. And the final case I was considering, that's why I circled and erased this first. This sort of cusp I guess, I don't know. That vertical little point there, and this one right here. I thought maybe those um derivatives. They might not have derivatives. And then I tried to figure out why I thought that and I couldn't come up with an answer. So probably there is a derivative there. 'Cause it turns sharply doesn't necessarily mean there's no derivative. So the main one where I think I can concretely say it doesn't have a derivative is at the hole at one, ah one half comma one. And if that's the case then it probably, then I don't know what to expect about this one because this one's at the same point. Or actually it's not. Anyway.

*Problem 10*

- Find the slope of the tangent line to the curve. (pause) Um. Okay. (very long pause)
- What are you thinking?
- Oh, I'm just trying to figure a way to get this in a form where I can figure out the curve.
- That's why you're doing this?
- Yeah. But I just figured out what I was doing wrong. I was rereading the question. It says find the slope of the tangent line to the curve and I got looking at this probably going to be something there. Like some sort of circle. And if I could figure out the equation for the circle I could draw the circle. And I could maybe figure out the tangent and the slope. Um. (erasing) Let's see. x squared y squared.
- What made you think it was an ellipse or circle?
- Well the x squared and the y squared. Ah. x squared plus y squared equals one is common for a circle. But this has got other components in it, so maybe it's perhaps it's an ellipse. I don't quite remember my formula for an ellipse but ah, let's just see here. x squared y. I want to put it in a form that. (pause)
- I don't want you to get too off track. If you try to do that you're going to get really . . .
- Screwed up.

- Yeah. It won't be easy to do. So I don't want to see you struggle through something for awhile and find out you don't get anywhere. Do you know of anything else you might do?
- Find the slope of the tangent line. Um. No. I know that somehow. (pause) (tape runs out) (mumbling) Oh. Maybe I could put it in a general equation. I don't know if this is right either, but the general equation for, what is it, what's the general equation for, something. Anyway. (pause) m x (mumbling).
- You're trying to put it into some kind of familiar form, is that right?
- Yes. That's exactly what I am trying to do. I'm trying to . . .
- You said something about m x.
- Yeah. Like I'm trying to look at this thing here, this blob of variables and stuff and I'm trying to say ah what can I do to this to make it something I can understand so I can get the graph and one point on it so I can figure out the tangent. But I'm so far . . .
- It's not going easily, right?
- I'm not figuring out what familiar form to put it in.

*Problem 11*

- (mumbling while reading question) (long pause) What exact point in time was the number of elk increasing most rapidly? And I put 1980 to 81. But I guess it's not the exact point in time. Or is that satisfactory?
- Can you tell me why you've put that?
- Oh. Well. It might help if I'm not reading the wolves, wouldn't it. I'm sorry, I screwed up. Elk increasing most rapidly. That would be (pause) between 82 and 82. That year in there. That one year span because the elks went from roughly just over a thousand to just over 35 hundred. And there's no other point on the graph that we see that increased that rapidly in that exact period, in that exact point in time. See now it's increasing very rapidly come 1985 through 1989. It's gone from 2 thousand to 4 thousand. But that's in less period of time. And it's still you know like you can't see. But right here, this space right here, that's when it increased most rapidly.
- Okay. What about part b?
- (long pause) 84 to 85.
- Why is that?
- Because the graph indicates that the number of elk decreases in this time frame from roughly 35 hundred to 2 thousand. Little over 2 thousand. Which indicates that the rate of change of the number of elk decreased. And there's no other point on the elk curve, (pause)
- What are you thinking?
- Oh, wait a minute. Ah, I was just looking for straight decrease. It says here the rate of change of the number of elk decreasing? (pause) (mumbling) (long pause) Hm. It's hard to say. I'd maintain the answer.
- Okay. Let's look at the bottom one.
- At what point in time is the number of wolves not changing? (pause) Number of wolves not changing. (pause) In the middle of 84 to 85. There's like this plateau, like where it evens out there where it doesn't change for a little bit. So that would be between numbers (pause ) Oh, and I see what you're doing here. You've got one, two, three, so maybe you wanted an exact point. So I'd say.
- That different time scale was for these different questions that I didn't ask.
- Oh, Okay. (pause) Um. I'd say right there. Between 84 and 85.
- Okay. Anywhere else?
- (pause) Maybe up here a bit. It didn't change between 86, it did, but minimally, little after 87. It's relatively constant.

*Problem 12*

- Differentiable. (long pause) Um. I don't know how to draw a derivative of 2 when x equals negative 5, but I'm going to take a guess. (long pause) Zero. (long pause) (mumbling)
- Can you tell me what you're thinking as you're going along?
- Alright. Um. I'm just graphing points and I was just, the two that are really providing problems is b and d. b is a derivative of 2 when x equals negative 5, and a derivative of zero when x equals 2. And I have no idea.
- Okay. Can you do any of the others though?
- Well, um. Local max is like is shown by just going up to that. So I'm not sure that I'm drawing this right in the first place. Derivative of 2 for negative 5. Local max is when it goes up like that, and then it says, and then I'm not sure what to do with this. So I'll just go like this. Then it says here slope of one when x equals 4. Um. (pause) You'd need it to go one over (long pause). I have no idea.

Personal Interview

- What are your reasons for taking calculus? You said something about you thought it would be easier than French.
- Well, yes. And also, ah, for any course that you need nowadays you have to take calculus. Like I know definitely in the sciences. And in the arts, for a lot of them, for economics and stuff you have to have calculus. So it's a good course to have if you can get through.
- If you didn't have to take it would you take it?
- Not a chance. I wouldn't go near it. 'Cause I know that it's not me, like after doing those questions I feel really stupid because I didn't get one of them right, you know. And I don't feel smart, and I don't know if that's because I'm dumb or because I'm not really good at calculus. Like I don't know, calculus.
- Alright. Do you see it as useful for you in particular then? You've basically already said that it must have certain uses by different people in society. But what about for you and your career?
- To be honest with you I really don't think, if I get my credits in calculus at university, and I eventually have to take other math courses, I can't see myself using my calculus knowledge later on in life because I don't understand it well enough to get any real use out of it. So to me I don't think calculus is incredibly useful. But I'm not the people that judge whether it is.
- Before you started taking the course were you apprehensive about it at all?
- Um, I never took Math 31 in high school and he had said that that was a prerequisite, or that it helps out a lot. And I hadn't taken it, so I was nervous about the fact that I was behind the other people. But for most people in the class it doesn't seem to matter. But yeah I was apprehensive because I really want to get my credits in it and not fail.
- Has that changed? Do you still feel apprehensive about it?
- More so. Because now I've, like I failed the first two tests of the year. And on the second one of the year I thought I did actually quite well on it and I did very poorly. And um, um, if I fail it I'm going to evaluate whether I'm going to repeat it or whatever.
- So you're not doing very well if you didn't pass either test, right? Um. Do you find it more difficult than your high school? Did you do well in high school or what?
- I'll just give you a brief history of my math.
- Okay.
  - Up to grade 9 I doubt I ever had a mark less than 9 in math. I was very good at it. Grade 10 I got a 68 or so, 69. In 11 I fooled around a bit and I got a 51. In math 20. And I didn't take any make-up. I felt like I didn't know graphs, like the concepts, but I didn't go into 23, or a 33 program. I went into Math 30 and I got a 70. So I pulled it

up. 70. And I think the class average was like 50 something. So I did, well I was quite happy with that grade. But it wasn't really, I didn't find Math 30 difficult. I just found it a lot of information. But I felt that if I concentrated I could grasp it. Does that make sense? Like ah I thought I could understand it and with calculus, there's like a lot of information and I don't understand it. So I just try to memorize important stuff that I need on the test.

- And is that working for you?
- Well, um, I can do, well like the last couple of tests I tried to understand it. Like that's what I'm doing for this test coming up, on Thursday. I'm going to go through all my theorems and everything that's been given us. And I'm going to go through the practice exam and I'm going to do all the types of questions he could ask. And I'm going to learn how to do those types of questions. And if, because last time I tried to understand it I just confused myself more because I don't have, like somewhere along the way I lost some kind of basis in math that other people seem to have. Because to me it looks like Greek on the board when he works through the stuff. You know, I got 8's and 9's in my other course and I'm failing calculus so I don't think it's the fact that I'm just stupid.
- Could you say more about what are the things then that are really confusing you? You have the general sense you are not understanding. Do you have a sense of what's causing that? Is it the amount of information, or is it the kind of language they're using and all the formality, or what is it?
- Um. Part of it I think is when they use, ah, what do you call it? They have a way of showing theorems using a and b and what's it called again? Ah, um.
- You mean the general algebraic form?
- Yes. Yeah. I find that kind of difficult. Especially when they do proofs in algebraic form. It looks like a whole bunch of b's and a's mushed up together. And another thing that didn't help was I didn't have my glasses. What do I feel is basically going wrong?
- For you.
- Um. Well I think somewhere back along the way somebody didn't explain it to me or I didn't pick it up. Some fundamental concepts. One thing that I remember is grade 10, the first test I ever failed was my graphing test in math and I got a 40 something on it. And I never bothered going back over. I did my corrections and everything but I lost something because I never did good on the graphing test again in grade 11. I did better on it in grade 12. And now this is like an important part.
- How much time would you say you're putting in studying per week? You have your hours in class, but outside of class in an average week how much time would you be putting in?
- Ah, usually when I study for an exam I study all at once because, this may sound like a typical answer, but if I study in blocks all at once it doesn't seem to me to ah, unless I do it by repetition, I'd say I put in about, in all honesty, about an hour a week to two hours. When a test comes I put in anywhere from 12 to 15. I work through the night sometimes. Like the last test I worked, the last test I actually started early, and I worked about 6 days before. Up until the night of the test I'd put in about 5 hours. And the night of the test I worked until about 2 or 3 in the morning from about 3 in the afternoon.
- And what do you do with your time?
- Well, I first of all start by going through the textbook and my notes and making up another set of notes and like figuring out all the different things we've taken. The theories, the, oh what do you call it, the theories. Like see, I bet I would do a lot better on those questions you asked me if I'd gone through and like studied my limits notes. Do you understand what I'm saying? Like ah, like a test I do that. I am prepared for that. So I go through all my theories. But then knowing them isn't enough. You have

to know how to apply them. So then I spend the other half of my time going through examples and practice exams and my assignments, and that's how I study.

- Okay. When you run into a problem, what do you do?
- Um. Well I work on it 'til I get it, usually. Or one thing that I do and I don't think that it's very good, like a lot I try to find somebody that can help me. Usually that's kind of hard to do, especially if you're studying. I don't know. Seems to me that I'm not the only one that's in the dark about calculus. There's a few people that get it, but a lot of people are floating around in the middle range and they don't really understand it. But ah, if I don't get it sometimes what I do is, if I'm working on the practice exam and I come to a question I just can't get, I look at how he has done it in the answer book, and I look and I'm like hm what did he do here. And I can't figure that out, you know. So I've got the theory right beside me and I'm trying to figure out what he's done here in this equation. And then once I think I understand it I put it away and like I've seen it all so I know exactly what to do. It's just a matter of writing it out again. But that seems to help.
- When you're working with calculus problems or ideas how confident do you feel in what you're doing?
- I don't. I don't. I could study for, like every time I think I did well. Like on both those last, like on the first test I didn't study very much. Like I think I studied four hours. And I didn't feel confident at all and I got like a 47. I came very close to passing and then the mark got scaled a bit so I did. And ah on the second test I'm like confident. I studied hard for this, I'll do good.
- And it was a long test too, wasn't it?
- Yeah it was. And he scaled it. My net score was a 37. 37 percent. And I was like "Oh man." That was on limits. And as you can tell I didn't do very good on your questions there. But then I went to a, then he scaled and I got a 46, or a 45.
- Well, in what ways do you decide when something is a right way to go about it? Or is the right answer?
- I don't understand.
- Um. Well, I guess what I really want to get at is are you willing to accept the things you do, or do you want something external telling you it's right?
- I like something external telling me it's right. In math I like multiple choice tests a lot better because at least, sometimes if you're working through it and you look up and see your answer there it instills confidence in the fact that you've got the right line or something. Or if worse comes to worst you can sometimes take the answers and work through using the answers backwards. Even though it takes a lot longer. But you know. That instills a little more confidence in you than it does having a page say, "Explain this function." Well what the hell do they mean by explain this function? Like to me that's so vague. So what, I don't know. It seems to me that obviously there's a variety of steps he wants you to go through, but where to begin and in what format to take confuses me. And I guess then yeah I don't have a lot of confidence in my own work because I know that I'm not good at it. And so I doubt my abilities to get it right.
- Well the language that's used in calculus, in your class or in your book, um and by language I mean terminology, the way things are described, the symbolic stuff. Do you find that helps or hinders you in learning calculus?
- Well, I don't particularly like reading the mathematical notation. I prefer to read the words that are printed out. And in the textbook and in my notes, well actually in my notes there's more mathematical notation. But he does explain a bit of it by words. But his notes are word for word from the textbook. So I just use the textbook. Um. And in class what I started doing the last week is following along in the textbook instead of writing. Because like um I'd always be trying to figure out what he was doing so I'd fall behind in my notes. And ah so now I follow in the textbooks and I understand what he's saying as he goes along. And I make notes in the margins as I go. And that seems

to be working better. I'm understanding it better. But yeah, I like the words better. The description.

- What does it mean to you to say you understand something?
- When I can a question set down in front of me, like identify this function and explain it and I can do it. When I can take any problem he throws at me on the test and I can come to a conclusive end with it. Or I can describe it in the way he wants me to. Ah, my level of understanding on it and his would be totally different even when I can do that. 'Cause all I'm doing is I'm doing is memorization.
- Is that to you understanding?
- Memorizing?
- Yeah.
- Well understanding, I get what you're driving at. Understanding is not . . .
- Well what does it mean to you is what I want to know.
- Well, it means two things I guess. In math it means being able to produce it on the test. And understanding to me though in the broader sense, English or whatever, means kind of "capiching" what they're saying about something and being able to apply this knowledge. But I've never been able to do that in math, and I, 'cause it's above and beyond what it seems like I can comprehend without, see like I got 5 other subjects right. And if I tried to go into that depth in math I'd lose my marks.
- What are the things you find easiest and what are the things you find hardest in this calculus course?
- Um. The abstract concepts like limits I didn't like. I find that hard. Um. Well right now getting into graphing with the a to h method or whatever. I find that that very involved to keep everything in line. I didn't mind stuff like derivatives.
- Do you find the textbook of use? Does it help or hinder your learning?
- I love it. It's great. It's saving me.
- Why?
- Well, 'cause it's ah very well laid out. It's a good textbook. It's simple to understand. They speak in laymen's terms. They speak so I can understand it. And ah that's good. And they provide lots and lots of examples and exercises which are useful.
- What about the lectures?
- The lectures. Um. I don't know. (pause) They're important in the sense that he goes through it, but I guess I would find the textbook more useful.
- Okay. Do you find the assignment exercises every week helpful to you?
- Um. Yeah. Yeah, I do. Um.
- In what way? Do they give you opportunities to (unclear).
- Yeah they give me opportunities to work with what we've covered that week.
- Okay. Did you find the tests gave you good feedback on where you were standing at that point?
- No. No, because I thought I was better at limits than that and I kind of blanked out on the test.
- What are other things outside that are influencing, I know you're in drama, but are there other things that influence your learning calculus?
- Yeah. Well, calculus to me ah, I put importance on it because it's my hardest subject, but in terms of my big picture I'll be very happy when it's over. So maybe sometimes it influences the way I think about calculus. And ah the fact is I guess, I have drama. And right now for example this month I'm at drama every single night. I had two tests last week and a term paper. I had an essay due this week and a bio report due today. An essay due at the end of the week and a test on Thursday. Plus drama every night. I've got dress rehearsal tonight and the show starts tomorrow. And I'm there every night 'til one or midnight.
- Are you getting exposed to calculus in any of your other courses? Science or anything?

- No. I'm in economics. Well I'm a polysci major, economics and I'm going to be majoring in political science, and then economics as a concentration. And English concentration.
- That's all.

Follow-Up Interview

- First of all you said something along this line. I don't particularly like reading the mathematical notation. I prefer to read the words in the book. Um. Or in class. And also that you like the book because they speak in laymen's terms. So I'm wondering if you can say more about that. What is it about the notation you don't like and why do you prefer the words over the notation?
- Well usually the notation doesn't really make a whole lot of sense sometimes. To say that a represents a constant, where I would say, well for example, like derivatives um like derivative is equal to f at a, minus, or f at b minus f at a all over b minus a. Um. To me, what does that represent? It's a little too ah, like it's supposed to represent that. To me it's easier if you just say that. Instead of writing it out with a's and b's. Like why don't they just say what they mean, you know? And ah I find that when they, like I always find myself looking at the formula in the big box, and then going "Well what does that say?" Like to me f at x equals this little slash mark times this little ditto mark or whatever. Well that doesn't tell me a whole lot. Thanks guys. And the I look at the explanation and the examples down below and that's where I get the true meaning of that. And then I can look at it. And then I can see it. But not, I can't work from the (unclear).
- Okay. So would you say that you don't particularly find that the equation helps you then?
- I find it helps after I understand the concepts.
- Alright. But it's not help for initially getting the concepts?
- Oh, not in the least. Not in the least. No.
- Alright.
- It just confuses.
- Okay. Well when you say that the book speaks in laymen's' terms what exactly do you mean? I got the sense that means you understand it.
- Well. Um. Yeah. Like. When I was little (tape runs out) And that was easy to understand. The reason they teach little kids like that is because it is something they can understand. And, I don't know, I guess I maybe need to look or something. 'Cause I still like being taught that way you know.
- You like the real world . . .?
- Well like, exactly.
- Kind of concrete examples?
- Yeah.
- Okay. Um. Go on.
- Yeah. Concrete examples. Um. The abstract, I like dealing with the abstract in English or political science or something like that. But the abstract in math is like (unclear) numbers all (unclear). That doesn't make sense to me. Math has got to be bang, bang, bang, you know. Logical.
- Well I did find that more than once you brought up examples like the apple. You brought up a swimmer and something with pencils at one point. You yourself brought into what we were doing last week. Um. I'm wondering why is it you do that? Is it because it does have more meaning for you?
- Well sure. If I'm sitting there with a pencil I'm trying to figure out like describe derivative. That's never been asked on a test. Like to me if I was to describe derivative I wouldn't explain it in mathematical notation. 'Cause mathematical notation to me means nothing. Because, to describe it in mathematical notation is just simply a

statement. That's all it is. I'd want to describe it in something else so that you'd have a relationship there so you could understand it.

- Alright.
- That's what I'd look for I guess.
- Okay. Um. This goes back to the textbook again. You said that you like the fact that the book has a lot of examples in it, and exercises. And they're useful. In what way do you find them useful?
- Well ah sometimes (unclear because of coughing) and even after the written explanation a concept is still unclear in like my mind or something. Like concepts like epsilon and delta is not easily understood. If I look at examples I can relate them to other problems. The formula is their way of expressing it. The explanation is my way of understanding it. The example is my way of applying it. Or at least, the textbook is set up that way so I can do that.
- Okay.
- Does that make sense?
- Yup, yup. That does. Um. When you're working through exercises what would you do if there weren't answers? Either in the back of the book or for your assignments?
- I don't think that would be practical. Because if there isn't answers how do I know if it is right or wrong? Unless they were in the back of the book. Well, okay each example is different. I find in math that I may know how to do a concept. I think I know. Then I get it on a test and there'll be a different, it will be the same kind of question, just a different way of doing it. And I will boggle it because this isn't the way I learned it. You know. And so I try to do as many examples as possible but I still think in terms of what I've done before. I can't in math I find it very hard to ah apply. Like in math because I find it difficult I try to memorize different ways of doing it. And if I come up against something that stretches my imagination and I have to apply myself in math I find it difficult. So if there wasn't like answers to go along with the exercises I wouldn't know how to do it right because I don't have a lot of confidence in my ability to get it right.
- Okay. So do you feel that you don't really judge it for yourself? You take that external thing that says yes you're right?
- Well if I do an equation and I come out and I go well this is my answer. Then I go and I look at the back and I've got it right. You have a good feeling when you get it right. Very very good.
- But you don't have that feeling before you look at the back?
- No.
- Okay. Um. Along that same line I asked you about how you know or what to you understanding is. Um. And one of the things you said was that when you get a test on the test you can do it and describe whatever he wants in the way he wants you to do it. Um. Do you see any difference between sort of the way you see things and the way say the teacher does?
- Yeah. Well yeah. I do because well he's a math professor. And I'm a political science major, university student. And I think the reason he's in math and the reason I'm not, like that in itself is the simple fact that we both understand things differently. He enjoys it. I don't particularly get off on it. You know. And also, what he wants and what I, the way I perceive it. I think you said it yourself. I think in our last interview. Something like, that was very, (unclear), like and the way I describe things is more along a different line. You know, than mathematical notation. So if I told him an example about apples or pencils, he'd just kind of go X. You know. It's just, like to him it's not what he wants you to know. And I have a hard time grasping. That's probably why I'm failing.
- Do you then see that your way of looking at it that way is somehow not as good?
- Well it's not.
- It's different, right?

- Oh, it's definitely different. But it's not, it's not, it's not mathematical. I mean it's just not.
- Alright.
- So that's not acceptable.
- What exactly do you mean by not mathematical?
- Not notation. Not ah, not described in terms of numbers and signs. It's described in terms of the examples that I relate to. That isn't mathematical. Math isn't, math is the study of number essentially.
- Alright. Um. When you study and you go through your notes and your tests, and um, and the examples and practice exams etc., what exactly does that entail? Like do you aim at trying to understand things or do aim more at memorization? You've sort of given me a sense you do both and I'm wondering what all that entails.
- Well, the last exam I was, trying, really trying to do good on this one because all I've got is the final. And if I don't do good on that I'm up the creek. But anyway, um, in that exam, I ah, what I did to study was I started off by looking at, I bought a book of practice exams from the bookstore. Where there's a practice exam and there's all the answers. I started by going straight to the answers and looking at the question and seeing what he did. Because I really felt that I had really no basis to start from because I didn't understand anything. So I looked through all the examples, exam questions, ah, find the critical points. All that sort of thing. And I started going through them, I learned how to do some questions. Then I started doing some of my own examples with the textbook at my side. And I worked through a few using what I had just read over. How to do questions. And then I did some more examples until I knew I was doing them right. Because I had just looked through it. Then I just flipped through my book (unclear) things I didn't understand. Then I read them again and again until I got the idea of what it was entailing. And then I did more examples of that kind of thing. And I got to the test and one thing that blew my test average away was that the one concept that I had in my book that was very difficult, probably not very much emphasis in the chapter, half the test was on it. And so I looked through it, a cursory glance, and I looked at it, and two pages, not even three pages. There was just this much explanation, couple of examples and that was it. You know. And I didn't understand it even after doing the examples. So I didn't spend a lot of time on it. And when it came to the test I didn't know how to apply it.
- Okay. Do you ever try to go through and when you're reading the book, and try to recreate it all for yourself, versus just reading it through?
- Yeah. I do but I try to remember practically. I don't really care if I ah can come up with my own ah ideas about what this concept means or something. I just want to be able to put it onto use. In all honesty, I'm not putting down calculus, it's not my ball of wax. If I can just get through the course I'm happy.
- Okay. Um. Well are you satisfied with the way of learning? You have said you do memorize it a lot.
- Yeah. In math I do.
- In math, yeah. I'm clear on that. Um. Are you satisfied with that? Is that okay with you?
- Well if it will yield better marks obviously, so I guess what I need to do is try to understand it more. But anytime I do that I just get so frustrated because it just doesn't make sense.
- Do you feel that if you had more time that might be something you could accomplish or not?
- Well time is a factor and if I had time, I don't know, it seems to me that if I had more time to look at it I'd get tired and lazy. I just don't want to do it. And I find myself coming up closer to the exam and I haven't put in the time I should be before the exam. Then a couple of days before all of a sudden something kicks in inside me, I can't describe it, and I do. And then two days before the exam I'll spend twenty hours. You

know. And I'll really give it my best shot. It's not really a question of time because I could make the time. It's just I don't.

- You're not that interested in it?
- You've got to have the interest in it to spend that much time in it. And unless it's out of sheer necessity I don't have any drive to do it. But before the exam I do get, like well okay (unclear).
- The motivation of having to write the exam, right?
- Yeah.
- Um. Well you did say you didn't see it terribly useful. Um. Do you think that relates to it?
- Ah. Like my, you know what my goals are. I don't see using it, like I see there may be business applications. Interest rates, maximization, whatever. (unclear) I don't see it as integral, you know graphing. Maybe if I was going into the sciences or something.
- Yeah. Okay. Well here's something else you said to me that you found doing these problems, not only were some of them different, that you felt that this um you couldn't remember things because you hadn't just done them. Um.
- Like some of these you asked me I remember seeing questions like that and I know if you asked me I can look it up and get the answer.
- But because, a lot of what you did was memorize, then you forget it easily?
- Well (unclear) the test. On the test I just kind of like, I usually on a math test I usually go to sleep after. Because I'm tired after it. And I don't know, maybe, maybe I just kind of let it, it dissipates or something similar.
- Alright. Um. On one of these (pause). You were working around with it and you were trying to put it in the equation of a circle or ellipse or something like that. And you said you were trying to put it in to a familiar form. You did that on a couple of the other questions I had equations written down for. I'm wondering if you could say more about that? Is that a way you try to approach many of these questions?
- Yeah.
- Like manipulate the rules so it looks familiar?
- When I was in physics. I don't know if this is going to be any help to you. But in physics it's all formulas. Physics 10 and 20 were, right. A lot of it was formulas. And that, I found that if I went in there and I just memorized it, on a sheet every which way I could twist round those equations, I did not bad. I think, I was fairly, Physics 10 report card and I figured out this way of studying. And I came out with a seventy-four in the course. I got, like I doubled my mark almost. And then in Physics 20, you know, I ended up with a 70 or 69. And I didn't like it enough to take Physics 30. But I always never found it really that difficult. Like it was actually relatively easy. I just didn't have an interest to put the time into it. And then you know Math 20, I almost failed Math 20. And I would say in Math 30 I just got my butt in gear and I got a 70 in that. And ah and um so when I say I put in a familiar form, it seems like that I've always done that. Like if I'm dealing with a graph and they say, say for example they don't ask for a graph, they just say what do these two points have to do, describe these two points. My first reaction would be to draw a graph, like plot them on a graph or something like that. Because I can understand that. That's a visual picture. And then like now I've got two things to go by. It's one more clue you know. And ah, if it's in a familiar form I can work with it because, this is rather simplistic, but I understand things like x squared plus y squared equals one. Okay I can deal with that.
- Because it's familiar?
- Yeah. Well it's like ah if I could take that equation and put it, objects in front of me and they said manipulate these ah so that you end up with, show how, say that was all in terms of food or something. Show how it all equals ah a banana and a pear. Well then I could work with it you know. So if I do the same thing with that I can ah, you know, work with it a little easier.

- Okay. Is that sort of the same idea when you're doing these examples? Like the pencils or a swimmer? Um. Because there was times it came up. You said you had your way of doing it and you'd say something like but that's not the correct way. Um. Something along that line anyway. Um. And I'm wondering how do you make use of your own interpretations and your own judgements versus other sources?
- Well in most anything else I could feel confident my views are um maybe not necessarily correct, but that they're feasible, or that I can show how my views and somebody else's views correlate or something. Like you know. In math I don't feel that I have got any basis to say that I'm right and I'm wrong. Because if they, they referring to math people, come up with all this stuff, or how do I say it. I'm just not confident that my way of viewing it, like it could so easily be wrong. Like I just don't feel I have it. A lot of (unclear) you know. (unclear) And it goes real fast. And I don't, even once for a minute, want to hold my ideas up next to somebody that has a grasp of it. Because I don't.
- Okay.
- So, I guess that's the way I can put it.
- Well then do you see calculus for you as being just a collection of these methods you do? Or does it have any personal interpretation? Or do you see that as somehow not important?
- I see calculus as a course I'm taking that will help me out among other things. And I don't see the ideas at all. I'm sure it will. And I um and right now it is just a matter of being able to produce it on a test. And whether or not my interpretation is correct doesn't matter. Because my interpretation isn't going to be counted (?) on the test. But if it helps me to understand it I suppose then it's valid. But it doesn't really help me to understand the notation. When I describe it this way. Because to me this is different from this. Even though maybe it's the same way. Do you understand what I mean?
- Okay. Anything else you want to say?
- No.

## Sally's Transcripts

### Interview 1

*Problem 1*

- Okay. If I were to tell someone what calculus is all about, I would say I don't really know even though I've taken half a year of it. And it's a very difficult math, and it doesn't deal with numbers as much. But it deals with ideas more 'cause it's all, you're always using, you know like functions and letters to represent things. Like in high school it was always using numbers.
- So it's not as much that way? It's more ideas?
- Yeah, more.
- Do you know what those ideas are? Can you sum those up in any way?
- So far. No. I don't know. That's a very tough question. What calculus is. Very tough. Did I answer it well enough?
- Do you have anything else you want to say?
- No.
- Okay. We'll move on.

*Problem 2*

- (pause) This limit, it will get closer and closer to four but it will never reach four. So the limit would be four. (pause) Um. That gets closer and closer to zero. No. Yeah. So um I guess the limit would be zero. But it would never reach zero.
- Okay, for both of these you said it gets closer and closer. Can you say more about that?
- Um. It gets closer and closer but it will never reach it. Because there's always lower numbers.
- Can you say more about either one? What happens as it goes along here.
- Well, here you keep adding one nine for each one. And . . .
- And what does that do?
- Makes it a larger number. This one becomes a smaller number.
- Will it ever actually reach zero?
- No it won't. If I, can I tell you this?
- Sure. You can tell me anything.
- I had this explained to me by my grade 12 teacher and it always stuck in my head. It's like when in the NHL, if a goalie has a perfect record, no goals he's let in. But then he lets in one goal. His record will never be clean. It will never reach zero again. That's how I'd explain it.
- Okay. Do you find it helpful as a way of thinking about it?
- Yeah, because when we were talking about (unclear), how it will never reach zero. But I could never understand how if it keeps getting closer it would never reach it. But that kind of explains it.

*Problem 3a*

- Okay, well using the rules that did, it would be like subbing in that. (writing heard) But then you just take the highest power, although, this is why I'm not doing too well this year I guess. Then probably the limit would be infinity. 'Cause this one would always be bigger than that one.
- You said you'd take the highest powers. Can you say more about that?
- Well in a (unclear) like in a polynomial like that. Like this is to the fourth degree. I think that's the term. And this is to the third degree. But if this was to the fourth degree, then you could just take the fraction of one to one. But it's not. It's to the third degree.
- What if it was to the third on the top and fourth on the bottom?
- Then this one would always be bigger, so then the limit would be zero.

- Now, what were you going to say?
- Just because when you divide a small number by a larger number it will always get slowly closer to zero as the larger number gets larger.
- Um. Now you said back at some point that this was a rule you were told. To you is it a rule or do you have a way of justifying it?
- Um. Well to me it's a rule but I guess it always works. Like he proved it does work so it's not just a saying, but something you can use to solve a problem.
- Do you remember how he proved it?
- If I thought about it and doodled around for a bit I could probably get sort of the (unclear).

*Problem 3b*

- (mumbling) (pause) Something that a number approaches, but it will never reach. Or something it can't cross like a border. Like you can't quite ever get to it.
- Can you draw a picture, or give me a hockey example? A picture or other examples?
- Well, I think of parabolas. Well you know how, you know how it goes like that. Well, not a parabola, but it would go. There.
- And where's the limit on this?
- Right here. Where it keeps getting closer and closer and I guess at the asymptote.
- Um.
- Is that what a limit is? You're not allowed to tell me. 'Cause now that I'm thinking about it, a limit, well he was saying (unclear) like this would be the limit because it approaches there.
- So where would the limit be in that example?
- x equals (unclear).
- Okay. Um. Do you have any other examples or ways of saying what a limit is all about?
- Um. No. Just something that something approaches.

*Problem 4*

- (mumbling) (writing heard) Well, I'd substitute 2 in for x 'cause it says that x equals 2. And then it will probably end up (writing) yeah, because this would be undefined. Or it's nonexistent because um zero over zero. So what can you say about the function? It either doesn't exist or it's undefined.
- What makes you decide that?
- Because it's divided by zero, but then I wasn't thinking, and I could factor that. So you have x minus 2 over (mumbling). These scratch out so x equals 3. No. At um when y equals zero, x equals 3. So y equals x minus 3. When x equals 2, y equals minus one. Does that make sense? So when x equals 2, y equals minus one.
- Can you explain to me what you've done?
- Okay. I factored it, and then um x minus two can eliminate because it's a common factor on the top and the bottom. And then so you're left with x minus three. So then I substituted in 2 for x because that's what x equals to. And 2 minus 3 is negative one. So then y would equal negative one when x equals 2.
- Okay, how do you resolve that, y equals negative one, with this zero over zero?
- I guess. I don't know. In class he told us when it's zero over zero it means more work. So I did more work because that just doesn't work.
- Do you have any way of interpreting this zero over zero? What does it mean to you? Other than the teacher says you need more work?
- Um. It means that the function (pause). Zero over zero, actually no. It doesn't mean that much to me except that it isn't, from first glance it doesn't look like it exists.
- Okay. But then this is what you get as your answer?
- Yeah, so it does exist after all.

- Okay. If I asked you to draw this, would you be able to either draw it or tell me what it might look like?
- Um. (pause) x. At y equal negative one, x equal 2. Something like that. I couldn't draw it just from that, but from using this. Then if y equals x minus 3 then I would interpret that to be a straight line. Then here's the y intercept at minus 3. The (unclear).

*Problem 5*

- Continuous or discontinuous. Okay, I like this kind of question. No calculations I think.
- You don't like calculations?
- Well, actually I love it. In high school I loved it. But in calculus . . .
- Calculations don't go so well?
- No.
- We'll talk about that more after.
- Okay. Um. This one is continuous because it never stops. There's no gaps. And there's no, like you could just keep drawing it forever.
- What about the others?
- This one's not continuous because you're here and then here you have to lift your pencil. And then you can go again. But it's discontinuous there. And this one, is that connected here?
- Okay.
- Then it's continuous. It's just not smooth. And it's continuous again for the same reasons. 'Cause you could just draw it and draw it forever and ever.
- What do you mean by draw forever and ever? Can you be more specific?
- Well, you can keep going and like there's no breaks. No breaks in the graph. No breaks in the function. The function keeps going. And this one's discontinuous again. Because here's the graph down here. And you can draw it here and then you have to jump again to here. And then as soon as you reach here you have to jump.
- If I hadn't given you the graphs. Say I'd just given you this one. Here's a function y equals x squared. Is that continuous?
- Yes it is because it's a parabola.
- Okay. If I asked you to prove it in some way could you? Algebraically?
- (laughter) Um. Algebraically, no. But I could draw you a picture of parabola. Oh. Algebraically.
- To do continuity formally, algebraically, could you do that?
- Like we do in class?
- Yes.
- Yeah, I could try, yes. Let me think now for continuous. How do you do that? (pause) Um. This is a bad day for math. (pause)
- Maybe if you turn it over so you're not looking at the pictures. y equals x squared.
- So what was that? Oh, I totally forget this. This f at x minus L. It wasn't that way was it? You can't tell me this can you?
- No.
- Okay. I guess I'll just say no I couldn't.
- You can't remember?
- No, I can't remember. If I had my notes I could.
- You said something about "as we do in class". Do you see that formal way of doing it as something you do in class and what I ask you here is somehow different?
- Um. I see what you're asking me as I do it. Like more, just the ideas of calculus. Like continuity. But then class, then you have the equations and method of calculations, and all that.
- It isn't just the ideas, or . . .
- It's putting the ideas to use. And here I guess it's stressing them.
- Do you find that putting them to use a stumbling block?

- Actually yeah. Because in class I (unclear) to question anything. But I understand everything, and I can learn it for the exam. I can even work on calculations and work through them on my own studying. But when I get to the exam I know what to do, but just the calculations just ah . . .
- It's what throws you?
- Yeah.
- Okay. So is it fair to say you find the ideas easier to get at, and get a sense of them getting the calculations all in order?
- Yeah. Yeah. But, the calculus calculations. This is just different 'cause there's no numbers that's why. Numbers are more concrete than calculating ideas.

*Problem 6*

- (mumbling while reading question). Um. A derivative is a slope of a graph. And when you have a function the derivative would be the slope which would be right here. And that's the way I would best explain it to anybody.
- Can you think of examples of derivatives?
- As in f at x being the function x squared. And then the derivative would be 2 x.
- Can you think of any sort of real world examples of what the derivative is about?
- No. No. I've thought about that actually.
- And you haven't come up with an answer?
- That's another thing about calculus that's so different from high school. In high school you do use it. But in calculus I just can't think of any real world illustrations of calculus.
- Alright. If I asked you to give the definition of a derivative precisely written down in algebraic or verbal form what would you put?
- (writing heard) f prime at x equals f at x plus h minus f at x over x plus h minus x which will equal f at x plus h minus f at x over h. And on a graph if you have a function and this is f at x. That's point P with x, f at x. And here you have Q which is x plus h and the function of x plus h. And then the slope is just the general formula y two minus y one over x two minus x one.
- Can you show me where that is on the graph?
- Where?
- How does this relate to the graph?
- Well, okay the derivative is the slope right. So the general form for the derivative would be the general form for the slope. So this equals this as in, well actually when you have a graph (unclear).
- You're marking them on the axes.
- Yeah. There's y two and there's y one. There's x two and there's x one. And then here's just the like here that would be x one y one. So x two y two. And you plug that in and you come up with that.
- Okay. Could you repeat that last bit?
- Okay. Here's the point rectangular coordinates of the points P and Q. And the points of P because it's the first point would be x one and y one. So x one equals x and y one equals f at x. And the point Q since it's the second point, x two equals x plus h, and y two equals f at x plus h. And then if you plug your point into the slope formula you come up with the definition of the derivative because the derivative is the slope.
- Can you mark where the slope is on that?
- Um. That would be the, and then oh I forgot. The limit as h goes to zero, because then what you're trying to do is um this is the h that you're adding every time here. Then um you want to shorten it so that this point slowly gets here. So that the function gets more and more closer to the actual slope.
- And where would the derivative actually be?
- The derivative then would actually be tangent to the first point. If you're trying to find the slope of the first point then it would be the tangent to the first point.

*Problem 7*

- Find the derivative of each of the following. Okay. These are long ones. Okay. 3 x. Okay, first of all I'll just, that's another one of my problems. I'll just skip steps and then I'll get off. I'm very messy.
- So where do you start on this one?
- Okay. First of all you put the derivative of y equals. Can I use f at x?
- Use whatever you want.
- (writing heard)
- Why do you prefer the f to the y? Do you have a reason?
- I guess just because I work with that more. And it's, when I see f at x I think y and function. Okay. So um. So now we want the derivative of x cubed plus one over x times the not derivative of the denominator. This is just the quotient rule. And then minus x cubed plus one over x times the derivative of the denominator. x squared plus 7. And that's over the denominator squared. Three x squared plus 7. And then you simplify that all out. So then the derivative of that would be 3 x squared plus, and then that is (pause) the derivative of x is one. So that should be just one. And the times x plus 3 x squared plus 7 minus x cubed plus x. And the derivative of that is okay um equals x to the half. So you get one half x negative one half. Plus x and the derivative of a constant is zero. Over that squared.
- I won't make you simplify it.
- Okay. So I can leave it like this?
- Do you find simplifying a difficult part of this?
- Actually yes I do. Very much.
- 'Cause you seem to go through this fairly easily.
- Yeah. This again is the ideas. It's know the ideas quickly jot down how to use them, simplifying is difficult.
- Do you find a lot of mistakes coming up when you're doing all that simplifying and algebra?
- Um. I think so. Maybe. Yeah.
- How about trying the second one?
- Okay. That's um (writing). So first you have the inner function. And then the outer function. So you get 10 (mumbling) times 4 t plus 3. And then 3 t, and then you use the multiplication rule for derivatives. Which is the derivative of the first one times the second term. Plus the first term times the derivative of the second term. So then we took the derivative of the first term. And then there's the second term. And then plus and then this normal term. Times the derivative of the first term. So we first take the outer function. (writing) times and then the inner function. One quarter times t is (writing) (mumbling) minus one and that would be (unclear). Okay. Do I have to simplify it?
- No.

*Problem 9*

- (mumbling) Okay. I like these. I like pointing things out on graphs.
- Do you find this much easier?
- Yeah I really do. It's much easier. Okay. It doesn't have a derivative there because it's an endpoint and the derivative has to have the function (tape runs out). And the derivative if it's going to change shape, I mean change direction, it has to be smooth. A point at which the derivative equals zero. And then the derivative is, the slope is more than zero here. The derivative is less then zero, negative.
- But it's not doing that here, right?
- Well it is but, what's the reason now? This is too, but there's a point at which, I suppose right here it would equal zero, but it's not. I don't know. No. It doesn't equal zero. Like the tangent to this line. There's not really a place where you could put a tangent. It could be here, or here. So there's no tangent for the derivative to be equal to.

And then here there's no derivative because the function has to approach the point of the derivative. Um. The point has to be the same. The function has to approach from every possible way. I guess that's the same as the limit. And so there's no derivative there because here it approaches negative one and here it approaches one half. (unclear) negative one and here it approaches one half. And there's no derivative here either I don't think. 'Cause it's not smooth either. It has a point. Neither here. Because it's not smooth. It goes on to negative infinity and that goes on to positive infinity. And they don't meet up. So again it's not approaching the same point. And, that's it.

- Can I ask you how you see this different from this? Here you argued how it's not approaching the same point. So there's no derivative. What about down here because it actually does approach the same point? How are they different in that they both don't have a derivative you said. But you're giving different reasons? Yeah. Um. Here it's not continuous, but here it's not smooth. Here it's not really smooth either because it doesn't connect. But here it is continuous. It's just not smooth.
- Anything else you want to say about this graph?
- No. It's beautiful.

*Problem 10*

- Tangent line to the curve. At the point zero, two. So. (pause) (mumbling) When x equals zero that will be zero and so that. So you'd have x squared equals four which makes sense because negative 2 squared equals 4. But I don't know if that's talking about the tangent line. I'm trying to find out how to do this. (long pause) Um. Is there a time limit on this? (laughter)
- What are you thinking?
- I'm thinking that I should know how to do this. But it has escaped me. The tangent line. The derivative. The slope of the curve.
- Can you say that again?
- The derivative is just the, well the tangent is just the curve at this point. So here's the point on the graph and the graph would be x squared. Oh, I'm not even sure what it would be. So I'd want the slope at this point.
- And how would you get that?
- That's what I'm thinking. I just. Um. f of x, well, the derivative is the limit as h goes to zero of f at x plus h minus f at x over h.
- Okay. I won't let you get way off. If you, can you do it in an easier way? Yes, that's the definition of derivative and you can go through all that, but I don't want you to get bogged down.
- Um. (pause)
- Is it the derivative you want to find?
- Yeah. (pause) Okay. Yeah. It would be 2 x plus 2 y minus 3 equal zero. So then 2 minus (mumbling). So then what am I trying to find?
- What's this you've written here?
- Okay. Well. I wrote down what the derivative would be here. 2 x. I just took the derivative of all this but I don't know if that's right. And then the derivative of that, and that, and that term. And then I sub in x and y. But then I get zero plus negative four minus 3 equal zero. Which doesn't make sense because negative four, this would be negative seven, equals zero. So that's not right. So now I can't find the slope of the tangent line.
- Do you know what's going wrong? Do you have any sense of that?
- Um. Just, okay, I took the derivative of this. Like this function. But then when I plug in the point, the point coordinates, I'm left with no unknowns. And also it doesn't work.
- You get the negative 7 equals zero, but you know that doesn't equal, right?
- Yeah

*Problem 11*

- (mumbling while reading question). Um. At what exact point in time. Right. It would be when the derivative of the graph was, is the highest.
- Okay. So where would it be?
- Right along here. Oh, do you want an exact number? (pause)
- Where would it be?
- 15 hundred to 3 thousand. Is that close enough or do I need to be more exact?
- And what was your reason for deciding that?
- Oh, point in time. Okay, sorry. I'd say in 1982. 1982. Because that's when the derivative of the function is the highest.
- And what has the derivative got to do with it?
- The derivative is the slope of the graph, and so um it says right here that the graph shows the number of elks in the park. So um when the number is increasing over the shortest amount of time then the slope is going to be the most vertical. And when it's the most vertical it's going to be the highest. And there it's the most vertical so it will be the highest there.
- What about part b?
- (mumbling) Right here. 1983. Right in this area here it's decreasing. Because the derivative, the slope of the graph is negative. So during what time period would be (pause) from 1983 the early 1983 to late 1984. (writing).
- And what about part e? At the bottom.
- Wolves are not changing. That would be when the derivative equals zero. So then that would be right here which would be at 1982 oh about April. And here the derivative equals zero. So that would be well, June 1984. And right around here which would be about August 1984. And that's just because the derivative equals zero. Or the tangent line.
- And what does it mean when the derivative equals zero?
- That means that um that the function isn't changing. The slope is level. So if it's level that means that the number which is on the y axis isn't changing at all.

*Problem 12*

- (mumbling) A derivative of two when x equals negative 5. So then, the derivative is like a slope of two. So then (mumbling).
- So you went up two and over one.
- Yeah. And here I'll go down two and . . . And actually though that's just when x equals negative five, so (erasing heard). That's just the tangent line. That's not the function. So I'll make it smaller so it doesn't look so much like the function. Local maximum when x equals negative one. This is all supposedly one function?
- Yeah, but you can do with it what you want as long as it has all those things.
- Okay. So then I can just kind of connect them all later?
- Sure.
- Okay. Then (writing heard).
- So you're going to make the tangent be part of it now, is that it?
- Yeah. (mumbling). (writing) So then here's the local maximum. Okay. The derivative is zero when x equals 2. So when x is 2 this has to be a straight line. So (unclear) and then, that doesn't look straight. There. And a slope of one when x equals four. (long pause) (writing and erasing heard) I'm using up a lot of your eraser.
- That's okay. I've got more pencils.
- Okay. So then this would be e, and then when x equals 7, a point where the function is continuous but not differentiable. Differentiable. Okay. It doesn't have a derivative. Um. Then if it's continuous it can't be an endpoint and it can't have a jump in the graph. And okay then it can have a point. Okay. (mumbling). Like that. Okay. So here's the slope of one. Here's f. That's kind of a maximum except . . .

- You don't have to draw it perfectly. It's not like my graphs are perfect.
- Okay. Well then here you are.

Personal Interview

- What are your reasons for taking calculus?
- Well, for one, in high school, I loved math. I just really enjoyed it. So you know, I go to college, I'm going to take math. And calculus seemed like, you know everyone takes calculus first year. And also when I came here I had intentions of going into pharmacy and for that you need Math XXX and XXX. So that's the major reason I took it. But now that I've changed that I'd have dropped it, but I'm in it because yeah, I do still enjoy it. I still enjoy math. It's just a lot . . .
- So your official reason for taking it is because you were going to need it for a course later on or whatever.
- Yeah, but I've changed that now. Yeah, I changed my plans, like I never dropped it.
- Do you see it as useful to you in any way?
- Not right now. No. Because I'm here, and as useful to society, I don't see calculus. It's not something I use at all. In all my other courses I can see it's more easy to apply them (unclear). You can apply it in chemistry, you can apply it (unclear). But calculus, I don't see it.
- Even by other people if not by you?
- Um. By other people I suppose. Um. Calculus teachers. (laughter)
- Other than math teachers?
- Yeah. And (pause). That's about it.
- What about number 3. When you started it, were getting ready to start calculus, did you have any apprehension about it?
- No. None. In high school math it always seemed easy to me. I never did a thing and I got really good grades. So I expected this to be the same.
- Has it changed?
- Oh, yes! Oh, yeah. I work really hard and my marks are pretty bad. So now is when I'm really anxious. Not before.
- And what's causing that?
- I guess, I'm not meeting my own expectations. And in math that's a big thing for me.
- So you're not doing as well as you'd like?
- Yeah, I had hoped to. I didn't realize like what calculus actually was.
- How well are you doing?
- Um. I've got a 7 point 5. I don't know.
- How were the two tests for you?
- Um. Very difficult. See, that's frustrating. I came to that and I had studied, and I knew what I was doing. But then once I got the test I didn't seem to be able to apply what I knew to the question.
- You had hoped to do better? Or, you felt you knew more than that, is that right?
- Yeah, yeah, I felt I knew more than that.
- Okay. So, I am right in saying you find it a lot more difficult than you expected and a lot more difficult than high school?
- Yeah. Definitely. Not only more difficult, but totally different.
- In what way is it different?
- Um. It's a totally different kind of math. It's not, yeah again, it's not dealing with numbers. And his examples, they're not examples, I don't know, they're not from life. You can't find problems that relate to life. They're just questions with a few numbers and some letters and words.
- So you find no way of making it real to you?
- Yeah! Yeah.
- Do you think if it was done that way, and more accessible or real it would be . . .
- Yeah it would be easier.

- Alright, take a look at number 6 there. And I want your ideas on each of these things in terms of how they help and hinder your learning. Part e will not apply for this class. The textbook first of all. Do you use the textbook and in what way?
- Um. Actually yeah, I do use the textbook. I underline it and I do all the exercises in it. Well this is of course usually right before exams because I have no time to other than that. But the textbook it um, it just explains about what he does. They seem to be very identical. Like he reads the textbook, he comes to class, he tells us what he read. I don't find them much different.
- Is it helpful, the book?
- Um. I guess I wish it was more explanatory.
- In what way do you see it's not that way?
- Maybe it's not that. Maybe it's explanatory, but he's just very explanatory all the time. I think that might be it. So then when I listen to him in class and then I read the textbook it just does the same thing over again. It reinforces it, but it doesn't I guess make it more real. It doesn't apply to any life situation.
- What about the assignments? Either the exercises in the book, or the ones you're given weekly?
- The ones in the book I like a lot. Because they have the answers. And so you can just sit there and work it out until you get the right answer. And the assignments that he hands out, I find them very difficult. And I'll do them, and then I'll come to the help session and ask questions, and he'll go over the ones we have trouble with. And that's nice because then he explains it to us. But it's still, when he explains it to you and you write it all down it looks so easy. But then when you get the same question on the test and you have to do it for yourself, even if you've done it for yourself already, it's still so difficult to do it for yourself.
- Do you have any idea why that is?
- No, I don't. That's why I came here. I was hoping you'd tell me.
- I have ideas. When we're done we can talk about it.
- But other than that, I'm glad, I'm very glad he gives us those assignments. Because otherwise I just know I wouldn't do much in the course. 'Cause it's so hard to motivate yourself when it's frustrating already. And it is frustrating when you try to do something and you just can't. Especially, yeah, I don't know. And so this kind of lets us know, once a week, we have one due once a week. So once a week you have to sit down, you have to do the assignment. And then, I try not to look at the answers he posts, 'cause then it's just copying them. So I go to the help sessions, and then he explains them. And that helps.
- Okay. What about the tests? Did you find that they were good feedback to you in terms of what you were learning, or not?
- Um. I don't know. They were very difficult and the second one was really long.
- Right. He mentioned that. He realizes it was far too long.
- Yeah. So. But yeah they were very difficult. They didn't seem to be, it seemed to be that he took what we had learned a step farther almost. So it seemed a lot more difficult than what we learned in class. I guess, I don't know, maybe I just should have studied more and I would have done better.
- What about the lectures? Do you find them useful?
- Yeah. He's very (unclear). He does a lot of examples. And, they're kind of boring, yeah they are very, like I do like, they are explanatory and they do help.
- Well, how much time do you find you're spending outside of class actually doing calculus on a weekly basis?
- Oh, about, it really varies. It really varies. Um. For one, I'll admit if I have an exam the next week then I study quite a few hours. At least an average of two a night. But if there isn't, maybe 3 hours a week, 4 hours a week.
- Okay. What do you do during that time that you study?

- Um. Usually I rewrite my notes. That is how I study for most of my courses. I've always done that. To write all my notes so that I can get it into my head. And I read the book. I underline really important ideas and I do the exercises. And I do the assignments he hands out.
- So when you go through and rewrite your notes and underline things, um, are there specific things you're thinking or doing as you go through that?
- Um. I'm trying to get the ideas and the ideas into my head and how to use those ideas. And then I try to do them by myself. And the exercises from the book. And then also, I guess it's just memorizing rules. Though that's not what math should be, but it is a bit because you have to remember rules in order to use them.
- Um. What are your impressions of the workload in this course? Compared to your other courses?
- Um. Compared to other courses the workload here is fairly light. Very light. Because there's just the assignment once a week. And that only takes about an hour to work through. And then there's the help session for extra help. And other than that it's just your own studying which isn't much. Unless of course you have an exam, but every course has that. So yeah, the workload is very light.
- Okay. When you run into a problem or you have a difficulty when you are studying, what do you do? I know you come to the help sessions, but if you're working through an exercise, say tonight, and you run into a problem, what would you do?
- Okay, for one, I'm very stubborn so I'll keep at it for, I'll keep at the same problem for an hour, and I'll just get more and more frustrated. And then I'll finally get very upset because it's not working. And then I have this really handy thing that my cousin is an r.a. and she's a math major. So I'll ask her to explain it for me, if she can. Sometimes she remembers how from last year and she'll explain it to me.
- When you say you're stubborn and you work at it for this hour, what do you do in that hour? What do you try to do?
- I'll just keep trying to figure it out with the methods that he showed us. And if that doesn't work I'll just use the rules and I'll try and apply them and I'll just keep trying to use a different method to get the right answer. And if there is not the right answer, then you don't know if it's right or wrong. Then I'll just, I'll just leave it.
- So, what sort of helps you decide when it's right or wrong?
- Well for one, in the book they have answers. So then that's nice because then you know when you have it right. And then you say "Oh, that's how I did it, that's how I was supposed to do it."
- And if it's not, suppose there isn't an answer? How do you decide?
- I guess you just use common sense. Um. Although for calculus I guess you can't really use common sense. I don't find that anyways. So then, I guess I say it's got to be right.
- So do you have a certain amount of confidence in your results then?
- Actually, no. Not in calculus. No.
- And why not?
- Um. Because. It's really hard to simplify. And there's so much room for error. Because if they get really long like that one you had me do. To simplify that there's so much room for one error. But, it would be so easy to get a wrong answer. And yeah, and I've always throughout math, throughout high school, I've always done really silly mistakes in things like that. Simplifying.
- Okay. Well, what are the things in calculus, actually I missed one. Let's go back to number 11. When I talk about language I mean the terminology, I mean the language that's used in the book, and how the teacher talks, as well as all the symbols. How do you find those things either helping or hindering your learning?
- Well I find them helpful simply because he uses them and then you know what the book is talking about. And vice versa. When you read them in the book you know what

he's talking about, it supports each other. But um I don't use that terminology much you might have noticed when I worked through those problems.

- And why not?
- Because, I don't know. It's just, it seems like silly things to memorize. Just terms, when I don't know. I just refer to them as, as what I think of them as. Instead of memorizing terms.
- So when you see the terms it's somehow, am I right in saying, maybe could you just say more about that? I'm not sure.
- Oh. Um. No. Um. I don't know. I just see the terms as they're just names you know. If you don't call them by the right name they're still what they are and a lot of times I just forget what they are called. And sometimes it's nice to know the terminology so you sound like you know what you're doing.
- But you don't find it actually helps you learn it?
- No, I don't. No, I think it's not the terminology that helps. It's the ideas behind it.
- Okay. What are the things in calculus you find easy and what are the more difficult things?
- Um. Well, there's not so much that's easy to, well, okay graphs, you know. When you have me point out on the graph. That's easy.
- So do you like working with the ideas?
- Yeah. That's really simple because you have it right there before you. I mean you (unclear) if it's right there before you.
- If you know the ideas. Of course sometimes you don't. And you're not sure of something. But I find that a lot easier than the other question, you asked me. Um. What was it. The tangent to that one thing, and I just couldn't think of how to do it. So actually I don't find too many aspects of calculus very easy.
- What is it that makes it difficult?
- Um. I don't know. Just, the whole fact that it's not concrete, it's not something there. You know. It's just ideas.
- And do you find that makes it harder to learn?
- Yeah. Yeah, I think so. Just that fact that you can't see it in life. It doesn't seem real. It's just talk. You can prove it, but that's just talking you know.
- In number 13 here I'm asking what things do you find help or hinder your learning. What I'm asking there is not just the things in class directly related to calculus, but other things. You know, the fact that you may have a heavy workload, or working part-time. In other works in your whole environment within which calculus is one small part. What are the things that are either adding to or taking away from your learning.
- What helps me learn it for sure is just doing exercises. Just lots of examples and keeping on trying. And I guess also I am taking a lot of sciences, and I see a lot of calculus in Physics. And stuff like that. And what hinders is all that, like I mentioned before, calculus is one of my weakest workloads. So I spend most of my time doing my other courses and then I remember, oh yeah, I have to study for calculus too. And calculus always seems to be on the bottom of my list of things to do. Because it's never really due. Or nothing ever has to be done except studying. I wish, like the assignments he gives out, I'm very grateful for those. But I also wish he had like more quizzes. He doesn't have any.
- So you'd find it helpful to have more regular feedback that way?
- Yeah. Or maybe even more assignments. Like two a week.
- Any other sort of outside pressures? Or the way the class is set up or anything like that.
- No. I like the class. Another hindrance I guess, it's just I don't know. The book is the same as him you know. It just seems to say all the same things, so it's repetitive. I suppose that's good too. It reinforces, but it also makes you go, "Oh yeah, I've heard that before." Then you close the book and try to work on something and you realize you don't know it after all.

- I think I missed a question here. Number 10. What does it mean to you to say that you understand something?
- To understand something I need to know exactly what I am doing, and I need to be able to do it. And um, on a test I need to know (unclear). I need to be able to show that I understand it by applying what I understand to a question.
- So how do you decide when you understand calculus?
- When you can give me a question and I can say, "Oh yeah." And then I can figure it out and show you the answer and you say it's right. That's when I understand.
- When I say? But how do you decide when you understand it?
- Okay. When I do a question and I look at the book and the answers match. Then I say, I guess that's the whole way of understanding calculus for me is getting it right. But I guess for a lot of people you know it shouldn't be that. It should be just knowing the ideas, but for me it isn't.
- Is knowing the ideas part of it or not?
- Oh yeah, I'd say you need to know the ideas in order to get it.

Follow-Up Interview

- First of all you said you liked the exercises in the book because they have the answers. I'm wondering what would you do if you didn't have answers? Either in the book or for your weekly assignments? How would you handle it?
- Um. Okay. I might do all the exercises and then ones in the book I guess I would just assume I did them right. And be an optimist. Or the assignments, then I'd go to him and I'd ask him to correct them because you need some (unclear) of what's right and what's wrong. Like if you know what you're doing or not.
- Okay. So do you, are you able to judge then for yourself or not do you feel?
- Um. Sometimes. You can tell if it works out or if you have to graph something you can tell if it looks right. Usually.
- Okay. Well you said something, you said often things aren't confusing when the teacher explains it, but then you can't see it on the test. Do you have any thoughts on why that is?
- Well because when he's explaining that stuff and I don't know. It's just seems, no, actually I don't know why. I've often puzzled. Because even if you ask questions he'll explain things, and I'll know exactly how to do them. And um, but then I'll do it myself, exactly how he did it and it just won't work. Well actually maybe it just doesn't work for some equations. Maybe I'm just not distinguishing them. Distinguishing what equations are different and what I can apply what he said to what one. Does that make sense?
- Mmhm. In other words you're not, you don't always see the differences between some of the things?
- Yeah. Like some of the equations here will be different.
- And you don't spot that?
- Yeah.
- Alright. Um. You said that you find both the examples in the book and even the teacher very explanatory. I'm wondering what do you mean by explanatory?
- Um. I don't know. Very thorough. He says everything and you're not left questioning things. You know. And yeah, he just goes over things very well. (unclear sentence) And so most people ask questions (unclear). And I'm sure if you (unclear) if you went to his office and asked questions. And so, yeah, there's never any reason for you to not know.
- Alright. You said that when you're working with your homework and you have a problem you're stubborn and you just keep working on it. Um. And you keep working with the method, and sometimes you finally get it. I'm wondering what sort of thing happens? How come you finally get it? Is it just luck because you've tried everything, or?

- Yeah. Sometimes it is luck, and sometimes you go back over things and then you find that you have been doing it alright all along. All the times and then you, or sometimes you don't (unclear) you have one little mistake. (unclear) get that right and then you can pick up how to get it right.
- Well then what does "getting it" mean?
- Getting the right answer. Getting the right answer.
- Alright. You said that for certain things you can prove it but that's just talking too. How do you see the derivations or justifications as they relate to the math and your learning of it?
- Um. I don't know. I see them as (unclear) and I know that it works. But yeah, I don't (unclear) because well you understand it but I guess understanding it helps.
- Do you ever try to recreate it for yourself?
- That's what I'd like to try but usually it doesn't work. (unclear) just try to prove something. It usually doesn't work for me because I get all muddled up.
- Alright. You also said that understanding for math is getting it right. But you also added in that to get it right you also have to know the ideas. Well what is understanding then? I'll just re-ask the question.
- Um. Understanding is applying the ideas to get a right answer.
- Okay.
- Is that better?
- Well, it's what it is to you. There is no right or wrong. It's what it is to you.
- Um. Yeah. I guess, yeah. (unclear) And you need to know the ideas in order to apply them. And know what ideas apply in what circumstances.
- Okay. I'm going to give you a couple of things you said while working on the problems. You said things like "He said in class this," "We learned in class there's a rule that means this," um, or "There's a rule and it works and you just need it to solve a problem." I'm wondering do you see these things as rules, or do you see these things as um, what am I asking? Well maybe if you could say more about that?
- I think I know what you're getting at. What comes to mind is, yeah, but it's more, I don't know. I guess it depends if you see it as more psychological or more scientific. You know scientific, science has rules. Whereas psychological is all ideas and understanding and all that. And I, and I don't know, in psychology you see things for yourself. But in science you just follow the rules. And there's not, well there is proof, definitely, but you know, there's all rules about how things work and why things work. And I tend to see things in a more scientific way. But then in math I work with rules. And how things work and what he says I just take as how to do it you know.
- Okay. Are you satisfied with this, or do you feel there should be more of your own judgements in it?
- Um. In a way I'm satisfied with it because my own judgment like I'll just agree with him that it's right.
- Okay. So do you feel a need to convince yourself that it works?
- Um. Yeah. Which is why I study. Why I do questions and assignments. Why I do questions. To make sure.
- Okay. To make sure it works. But do you feel any need to convince yourself of all the background details of why it works?- Actually no.
- Alright. Um. (pause) More than once you said to me "Can't you tell me," or "If I had my notes I could do this." I'm wondering how do you make use of these other things? Either me, or the teacher, confirming it for you? Or the answers at the back of the book? Or um, what do you mean when you say if I had my notes?
- Um. I look at them to confirm it. To confirm that I'm doing it right and that I'm not doing it wrong. And I know that it's the right answer.
- Okay. Um. More than once you referred to calculus as being different from other math because it dealt more with ideas. Do you have anything more you might say on that?

- Um. Mm. Not really actually. It's just not too evident in everyday life I don't find. (unclear for a couple of sentences)
- Okay. Um. (pause) At one point you said to me you prefer having f at x equals rather than y equals. I'm wondering why.
- Oh that's (unclear) because f is, I always use that. It's no different, f and y. It's just well f at x is just more (unclear).
- Well then how do you see the notation and the language fitting into your learning?
- Ah.
- Does it have meaning for you?
- Um. The notation of the derivative? Like that?
- Any of the math notation.
- Well you just pick up on what you like and you're comfortable with all the time.
- Okay. Um. This limit here. What would you put for that one?
- Um. That it would be four.
- Alright. Could it ever reach four?
- (pause) Mm. No. Oh yeah it could actually.
- Why do you say no and then yes?
- Because (unclear). I don't know. I never really actually understood that. If a limit, it can . . .
- If you want to write something go right ahead and write it on here.
- Sometimes if you have a graph and there's a (unclear) here. And it goes like that and you (unclear) limit from both sides. But then sometimes also (unclear) then it went out. A parabola. It gets closer and closer to the line but never reaches it.
- So can something reach its limit?
- Oh yeah. It can.
- Alright. But in some cases, is this the case where it does?
- Yup.
- And what about this other one?
- (unclear) never quite understood.
- Alright. Um. I think that's about it.

## Jennifer's Transcripts

Interview 1

*Problem 1*

- I think calculus just seems to be breaking things down. Breaking like (pause). Breaking things into sections and then analyzing these individual slices or whatever.
- What do you mean be breaking into sections?
- Like um. (pause) Like a graph. You take each individual point on any sort of graph and you're like slicing up a function type thing. Analyzing each thing and getting a constant, I wouldn't really know how to explain it really. Um. (pause) Could it maybe be finding like an average? Like with these slices or whatever. You're taking the derivative but in calculus it's finding what that average between them is.
- Alright. Is there anything else you can say about it?
- Probably. Ah. Calculus.
- In ten words can you tell what it's about?
- Ten words. Calculus. (long pause) Mm. (long pause). Taking functions, algebraic functions. (long pause) Can't really. Okay. (pause) Hm. Let's see, what calculus is about. What's calculus about? Ah. (long pause)
- Maybe that's what it is to you, right? Is that what it is?
- No. Calculus always seems to be, to me anyways, graphical explanation of things that happened around.
- Okay.
- Like using an algebraic expression and finding a graph of that. And then being able to explain just what's going on, through the original function.

*Problem 2*

- Hm. (pause) (mumbling) Well I would say zero for the first one. And four for the second one.
- Okay. Can you tell me why you decided this?
- Just 'cause it seems to be getting smaller with bounds. That seems to be getting closer to four.
- Okay. Is there anything else you can say?
- Ah. (pause) Not really. Just that that would be how it strikes me.
- Would it ever reach zero in this one?
- Probably not. I'd say it maybe comes to a close point.
- But it wouldn't get there?
- Yeah.
- What about in terms of getting to four? Will it ever get to four?
- I'd say the same. It would probably get to a point very very close to four.
- Okay. Do you have any other way of representing this? Either algebraically or with a graph?
- Hm.
- Or can you just explain it? Is that all?
- (pause) With a graph? Ah. Just like for this one it would probably be like four there. Just like that or something like that.
- Okay. And what would happen?
- Ah.
- Can you explain how the graph relates to this sequence?
- Well if this was like a four it'd get closer to four, well.
- Okay. And what about in the first one? Anything?
- In the first one? Ah. (pause) Going from one to zero. So (pause) they all seem to be with one decimal place. Is that how (unclear), maybe. I don't know how I would really describe it.

- Okay.
- It just seems logical. I don't know.

*Problem 3a*

- (writing) I'd say it was undefined.
- Can you explain to me what you've done?
- Um. I've taken the highest power. And I've used it, oh, sorry. I didn't, I think I did this wrong. I didn't take the highest power there. Oh, I'd say it's one. Taking the highest power in each. In the numerator and denominator. Highest power of the variable and dividing it through.
- Okay. That's what you've done here.
- Yup.
- And then what did you get?
- I got one plus zero there. And then one. (mumbling) And then it would be one over x squared and five over x cubed. So those are two zeroes. So it would be one over one.
- Okay. And why is it you did this very first step? You took this and you divided it by x to the fourth. Why did you do that?
- Just ah. (pause) I don't know. It's just something I've been taught.
- Okay. You don't know why?
- Where I would derive that from, no.

*Problem 3b*

- What does the limit mean to you? As the numbers get larger without bound it would approach one.
- Okay. In general? Like either with this one or the others I gave you?
- What does a limit mean? Um. (pause) Gee. Let's see. As a number, the variable approaches a number, (pause). Ah. A function could equal a number, maybe.
- Can you write that down?
- So, okay. The limit of x is going to one of a function of x. It would be equal to two as the x in that function is equal to one. Or getting close to one, the function will be equal to two.
- Okay. Could it ever equal two?
- No. It can only get so close to one.
- And not actually ...
- Yeah not actually reach it.
- It can get so close to one?
- Yeah.
- What about two?
- What about two?
- Can it reach two or can it only get so close to two?
- Oh, it can get to two.
- Alright. You're not sure?
- This can't get to one. No I guess you couldn't get to two because it wouldn't actually be an exact approximation. This is just getting so close to one that you'd get so close to two.
- Okay. But you wouldn't get . . .
- You wouldn't get to two. You wouldn't get there. The limit is equal to two.
- Is that always true in all limits? Can it ever actually reach whatever the limit value is?
- Yeah.
- Okay. Let's go on. Unless you have something else you want to say.
- I was just thinking infinity, I don't know what that would be. But. Like if you have a limit as x approaches infinity. I don't know how that works.
- Then what? What were you thinking about that one?
- Well. You can't really approach infinity. Or, I don't know.

- You sense there's something different with that one, right?
- Yeah. I see it should be different.

*Problem 4*

- At x equal to two. You just equate it. (writing) So it's undefined. Or. (writing) It's negative one.
- So what could you say about the function?
- The function. When x is equal to two the function is equal to, y is equal to negative one.
- Okay. First of all you said it was undefined and then you decided (unclear). What went on in between?
- Oh, I ah I factored it. I factored it. 'Cause it just, it looks factorable. And then you just cancel.
- And you get?
- Negative one.
- Do you know what the function might look like around there?
- At y equal to negative one? Ah. It'd just be a straight line. Would that be where ah (pause) I don't know. y equal to negative to one. Then x would be equal to (pause) (mumbling). Would you equate negative one equal to this? Equal to that function?
- You tell me what you think.
- Ah. (writing) That would just be. I don't know. I'm running into circles here. Okay. At y equal to negative one. The graph of the function. So these would be (pause). (mumbling) It would be a parabola.
- Why do you say that?
- Just, to the second degree.
- Okay. You think it might be a parabola, but you also said at one point it might be a straight line.
- I was looking up here. I didn't think. To equate y equal to one. I was just looking at x equal to (unclear) y is equal to x minus three. But if you put y equal to the function back in there. I think it would be a parabola.

*Problem 5*

- (pause) I'd say hm, these two are continuous. This one is discontinuous. 'Cause there's a hole in the graph. This one. Hm. (very long pause) Actually this one is discontinuous too.
- Can you tell me your reasons for these?
- Okay.
- The first one you said was continuous.
- Yeah. Because. (pause) If um. Gosh. I can't seem to back up my things here. (pause) Because there would be no point where the graph would be interrupted.
- Can you say more about that?
- Hm. Ah. Maybe I'm not quite sure. Like if it's continuous or discontinuous. Continuous would be meaning that the graph, well we were always taught that if it's continuous you don't have to lift the pencil from the paper. That there would be no breaks in the graph. And this would just continue up without end. With no breaks. And same with this one. It would approach the x- and y-axis with no breaks. But this one it's got a break in the graph. At x equal to one. And this one. Okay. (pause) This is like an absolute value type thing. It would have to be, it would have to be, not an absolute value, like it's a condition. (pause) So that one, x is less than zero, it's one at y. And when x is greater than zero, or x has to be greater than zero beyond (pause). Hm. Wait a sec. (mumbling) (long pause) Oh. y is equal to one if x is less than zero. So that would be there. And y is equal to x plus one, x is greater than zero. So whatever this one is the slope would be increasing that way. Like y is (unclear). This would just be a straight slope up. I'd say it's continuous.

- Okay. Do you have a reason for that?
- Like before, there just isn't a break in the graph.
- Alright. So that's your way of expressing whether it is or isn't continuous. Do you have a way of formally writing that down to justify it? If I gave you this one, we won't take a hard one, y equals x squared. And asked you to prove it was continuous algebraically, could you do that?
- Could you take the limit?
- Is that what you'd do?
- Yeah.
- How would you do it? Could you write something down?
- Ah. Take the limit as x, let's say we have a positive or negative infinity. Or any number within the real bounds.
- And then what?
- And then, well. (pause)
- Go on with what you were going to do.
- Okay. And as the numbers get larger in x there's nothing that would. This is how I'd usually do it I guess. So it would be equal to one as the function of x is equal to one. So it will just keep on getting closer to one. And there will be no break in the graph to stop it from getting closer to one as it approaches. That just seems to make sense.
- In what way does it make sense?
- Oh it just means that this graph is exceeding one.
- Okay. (pause)
- So. I imagine you can just, using values in there, there'd be no values of x where the function wouldn't exist.
- Alright
- Like for any element.
- Do you have anything else you could write down?
- I'd just say any, it would be continuous because any number that x, the function would exist at any number x.

*Problem 6*

- (long pause) What the derivative is all about. Hm. (long pause). The derivative would be the slope of any line (unclear). Can I just say this?
- You don't have to write. Can you say more?
- That um, when I think of derivative I always think of it in terms of graphical.
- Can you show me that then?
- It's just, well I'd just say that, I mean like every term that I've ever used derivatives in or that I think of, I think of f prime being a way to solve the graph of a function or something like that.
- Okay. Can you give me an example or say more?
- (writing) Okay. f x whenever x plus, I don't know, x cubed plus 5 or whatever. f prime would be just 2 x plus x squared, (unclear) x squared. Now that would just be the slope of a graph and you just plot the graph or whatever. I've always seen derivatives as a way of expressing an algebraic expression in a graphical sense.
- Could you do that? The graphical part?
- Of this one?
- Sure. Probably not. (mumbling) Kind of incoherent writing. I don't even know if this would work. But ah. It's quadratic. (writing) Then. (pause) I'd do an interval test or something. But. (writing) So it's increasing all the time. So. Or according to mine. (pause)
- Does that help you draw it?
- Not really because it seems like I'm kind of contradicting myself. I found it to be increasing for all the time in the intervals.
- How does that contradict yourself?

- I don't know. It seems that it'd be parabolic. Maybe it's a certain point where it's like that. And I haven't found it.
- Alright. Um. Since you seem to be having a bit of a problem with that one, can you take, just draw me any graph, right down here. And tell me how you would see the derivative. (tape runs out)
- Okay. Well we can just take like maybe a normal line.
- Sure.
- Sure. So ah. y is equal to whatever. x (mumbling). And the y of that function of x prime would just be one. And the slope would just be, that expression could be changed into something graphical as being the slope of this (pause) would always be one.- Okay. Um. Well anything else you'd say to your friend?
- Well the derivative. Hm. (pause) It also seems to ah (pause). Maybe it expresses magnitude. It seems to always be expressing things, like a rate of increase or a rate of decrease.
- Okay. Can you think of examples of that?
- Well related rates or something like that. It can, it's something that can, you can relate ah, well you know how velocity is a function of x. Or a function of t. Velocity is f of t or whatever. It seems that it's able to express things like if they increase. I'm not sure how I would really say that. It would have a rate (unclear).
- Okay. Do you have any formal way of representing a derivative? Algebraically writing down a definition, or showing me where it comes from?
- (pause) Like what the formula would be?
- Yes. You can use the back of the page if you need to.
- Like just maybe the general formula x. Plus one, like that?
- What's that you've got?
- The definition of how to find the derivative.
- Can you say more?
- Hm. Ah. I don't know really what you'd want me to say, but.
- Well what would you do that for?
- What would you use that for? To find the slope.
- How would you use it? You've written something down there. I guess what I am asking is what does that mean to you?
- What's this mean to me?
- N x to the n plus one?
- It would be a definition of how to find the derivative of any algebraic function. Like x to the third, and 3 x squared. Maybe I'm not really sure how to explain this, obviously.

*Problem 7*

- Can you tell what you've done now? And what rules you've used if you know.
- I've used the quotient rule.
- And did what? Maybe just outline quickly what you've done.
- Okay. What you're saying is the denominator itself is an entire variable. Taking, left the denominator and found the derivative, I mean left the numerator, found the derivative of the denominator. Minus the derivative of the numerator times the denominator. All over the denominator squared.
- Okay. How about the next one?
- (writing) For this one I would use the inside outside rule. And the product rule. The inside outside being the exponent, times the exponent in front of whatever is in the brackets. Minus the exponent by one, and then find the derivative of what's inside the brackets. And then using the product rule. The first term, prime of the first term times the second term. Plus the first term times the prime of the second term.
- Okay. If I asked you to simplify that could you do that?
- Probably.
- Could you start? We won't necessarily go all the way?

- (writing) I'd probably just take that ten out front. (writing)
- Okay. I just wanted to get you started. What were you going to say?
- Nothing.

*Problem 8*

- That's the definition of the derivative using limits, isn't it?
- Okay.
- What interpretation would I have?
- Do you have any way of saying more about it? Or showing me where it comes from?
- Um. Well there's that graph. As well, whatever. And this would be x. And then, x, and x plus h. (long pause) Hm. H approaches zero. (long pause).
- What are you thinking?
- I'm not really sure. It would have to, I'm trying to think of, as x approaches, as h approached zero. (pause) So is this. The distance between here. h approaches zero. The limit of that would just be the slope of that tangential line there. This would be getting closer to that. These lines would be there at that point f prime 'cause that's the definition of the derivative. Would be the tangential slope at that point.

*Problem 9*

- (long pause) At ah, (pause) here I would say. And maybe at the asymptotes because where the function of x, oh I guess not. (pause) The function of x does not exist where the vertical asymptotes are. So at five the function wouldn't exist. And at one half it wouldn't exist. I'm not sure how you would relate that to the derivative. If the function doesn't exist. (pause) Maybe the function wouldn't exist, the derivative of the function wouldn't exist where the function wouldn't exist.
- Does that make sense to you? Does it seem reasonable?
- It seems reasonable to me.
- Why?
- Well if the function weren't to exist there would be, I couldn't see how the derivative of that function, it would be the derivative of a nonexistent function.
- Alright. And that's what's happening at the places you've circled?
- Yeah. At the actual vertical asymptotes.
- And what about at this one?
- It wouldn't exist there because at that point as it approaches it will never actually get to that point. But it approaches it, so it can't equal it. It wouldn't exist at that point.
- What is it again? Can you just say it again?
- It would get to a point where it could get close to that point, but it would never be able to equal it.
- The function wouldn't?
- No.
- Are there other places there wouldn't be a derivative?
- Hm. (pause) Somewhere around here maybe. I'm just trying to decide that. At (long pause). I'd say right there. 'Cause isn't that, I forgot the name for it. Like ah, (pause). I don't know what it is.
- How would you describe it? Down at negative five where you've circled?
- (pause) Um. (pause) Well the function is has, it's changing there. Um. It really can't continue there so it has to change there. So there can't be, it has to change, the function has to change at that point there 'cause it can't continue with the way the function was. So it has to make a drastic change.
- Alright. And how does that affect the derivative?
- If the function is unable to continue like that then I'd say the derivative wouldn't be able to.
- Alright. You also circled on the left here.

- (pause) Same thing like well there it's not allowed to continue. That's the point where it begins. So it's not allowed to continue off beyond that. This point (unclear)
- And how does that mean there's no derivative?
- Okay. Let's see. I'll have to think about that one. (long pause) Well at that point it's got, it's just a single solitary point. It's not really a function. It's not like an entire expression. So it's like y equal to one or whatever. It's not a point. Or, it's just a point. It wouldn't have like a slope or anything like that.
- Okay. Any other points you want to say anything about?
- (long pause). Ah. Maybe that one.
- Why?
- Um. Well it's like that other graph there, or question or whatever. Five. Or six. That beyond this point it's permissible to have values beyond that point but the values have to change after that. So the function, this function on this side wouldn't be equal to the function on that side. The function wouldn't exist at that point. The derivative . . .
- Does that seem reasonable?
- Yeah. There seems to be a point where this one isn't, the function isn't equal to this function on the other side then at that point there it would seem discontinuous.
- And what about the derivative?
- Again I'd say if the function is discontinuous at a point the derivative (unclear).

*Problem 10*

- (writing)
- Okay. That's your slope?
- Unhuh.
- Can you tell me what you did very quickly?
- Very quickly. Yeah. I did implicit differentiation. So using the product rule for the first term found, the derivative of x times y. Plus the derivative of y times x. And then for the second term implicit differentiation of x y squared minus three. Just differentiate that. Differentiation of a constant is zero. And then isolating y prime. Take all the terms which don't have a y prime to the other side of the equation. Take out the y prime. And divide the factored, what's left of the factored form through the other side. You isolate y prime. Then just substitute the points in for x and y.

*Problem 11*

- Okay. (long pause) (mumbling) Would just be looking at it?
- What would you say?
- For a, I would say between 81 and 82.
- Why?
- Because the line is steepest at that point. And while it's increasing here it's over a large period of time with a slower increase.
- Okay. What about b?
- Ah. (pause) Between 3 and 5. 'Cause again the slope is going down.
- Okay.
- (pause) Number of wolves not changing? (long pause) Between 86 and 87 right there. It seems to reach a plateau in there.
- Right. How would a plateau indicate not changing?
- Well, the line isn't increasing or decreasing. It's just maintaining a steady slope. It's flat.
- Okay.
- Like this would be a steady slope up but that would just, that would indicate decrease. (long pause) Or could you take like that, the number of wolves at this time would be the number of wolves at that time?
- What would you say then?

- Just that they have increased to a point but they've also decreased equally in that space of years.
- Okay.
- Is that what?
- Does that answer the question?
- (pause) No, I don't think so.
- That's a good observation.
- Hm. Yeah, I would say 86 or 87.

*Problem 12*

- (very long wait) Okay. Let's see. Okay. I wouldn't really be sure how to do this, but the derivative would be equal to two. So the derivative being the slope. There's a local maximum at negative one. (pause) Okay. (pause) I'm not sure like how you'd find the equation of the line, or if you even need to.
- Do you think you need to, or can you draw me a graph that has these properties at these points and whatever else you want in between?
- But wouldn't you have to know what's going on between these two points? (mumbling) (very long wait) I don't know if I could draw a graph without knowing it. (pause)
- Can you tell me what you have done here?
- Oh. I was just looking at points when x is equal to numbers like the derivative of two when x is equal to negative five. You'd have a slope of two. So I was just kind of drawing it through those points. But it could also be like anywhere up there. You don't know what y would be. And the same thing for a local maximum at one. You know it would be like that.
- Alright.
- And a derivative of zero when x, it could be like there or whatever. You wouldn't know what the y value would be. or I wouldn't know what the y value would be. The slope of one, just slope. Connect the dots I guess. Number f, I don't know about that.
- Okay. What about g?
- (pause) It's greater than 8. (pause ) (mumbling) (long wait) At any point past that (pause) it would be just like a point of inflection? It decreased to a point and then at f prime it'd have an inflection point.
- What makes you say that?
- I don't know. Well if f prime is less than zero, a decreasing slope. But if f prime, f double prime is greater than zero, wouldn't that be. I was thinking intervals here. Maybe that's not right, but.
- What was it you were going to say about f prime?
- Oh I (pause) If f prime is negative it just means it would have a negative slope. It would be decreasing. And with the second derivative greater than zero it would be positive. It would be like concavity or whatever. So it would be up, concavity upwards.
- Can you put that together?
- It would just a point like maybe that.
- Alright. Um. What were you going to say?
- Oh if it was going to be the other way, but no.
- And it's part f that's really confusing you, right?
- Yeah.
- Do you have any idea what that's about?
- (unclear) those other questions like maybe the point is like this where it could be continuous or (pause) the function could be continuous but not differentiable. Like um (pause).
- Is that a case of it?
- I don't know.
- You're not sure?

- No. I was just kind of floating around. A point where the function is continuous but not differentiable. (pause) Yeah. I wouldn't really know what to do about that one.

Personal Interview

- First of all what are your reasons for taking calculus?
- Taking a calculus course? It's a requirement for what I want to do and also to get a good mark I think.
- If you didn't have to take it would you take it?
- Yeah. Actually I like math. I took 31 in high school and I like this.
- Do you think having taken 31 helps you here or not?
- Yeah. It does.
- Do you see calculus as useful for whatever your plans are?
- Oh yeah. Like in physics, in even stats, you use calculus for everything. I think it's useful.
- And what about in general? Do you think it's useful to society?
- Well I mean, using physics as an example, say like it makes these seemingly hard things very easy. Very easy to explain functions and stuff like that. Relationships.
- Before you started the course were you apprehensive about it at all?
- No.
- And why not?
- Because I had taken it before.
- Have you changed at all?
- I think. I'm finding it a little harder. Well actually during the first couple of months they didn't seem familiar to me, the things.
- Did that make you a little unsure?
- Yeah. It did. It made me a little apprehensive. Yeah.
- But in general how are you feeling about it?
- Oh. I like calculus.
- How well are you doing?
- How well am I doing?
- At this point?
- Good actually.
- How well did you do on your midterm?
- I got a 75.
- Okay. And what about your weekly assignments and quizzes?
- Yeah. Those are pretty good. Like both, lab quizzes and stuff like that?
- Yes. Okay. Are you satisfied with that?
- I'd say I'd be having a little more, if anything the only thing that would really giving me problems are the assignments.
- Why is that?
- Well they just, for one thing I don't really like the textbook that much. And they seem to ask these questions but they have no relation of how you would do them. Like in lectures the examples are quite equatable to what we do in the tests. Whereas the textbook would be a little harder, I would say, doing the assignments.
- What else can you say about the text? You don't like it but . . .
- It just doesn't seem to really explain using common words and stuff like that. It seems to be taking like too many theoretical things instead of making it a little more practical.
- So you don't find it particularly helpful?
- I don't find it helpful, no. The notes I find help a lot.
- From class?
- Yeah. I would be lost without the notes to tell you the truth.
- What's the difference between the text and the lectures that's making one better than the other?

- I think maybe the lectures it's taking a simple example and then expanding. In the textbook it's just laying it before you and kind of telling you to figure it out.
- Okay. How do you feel about the assignments? Are they helpful to your learning?
- Yeah. They are.
- In what way?
- For one thing it just gets you to do it. I mean it's very easy to sit back and say oh it's very easy. But when it come to doing something you have to rethink what you have to do. You go through a method I think of solving problems.
- And what about your labs and lab manual?
- I find that helps a lot. It does.
- In what way?
- It's very simple and it's just straight forward do this do this do this. And giving you guidelines.
- And you find that very helpful?
- Yes. Do you find if you don't have that, is it hard to sort things out for yourself?
- Not really. I find they are very easy. The labs. It doesn't seem to be straining your intelligence any. You know. Everyone seems to be doing well in the lab. I don't know if it's really necessary, but it does help.
- Alright. Um. Well how much time do you think you're spending on average outside of class time in one week doing calculus?
- Not very much I have to say. Maybe two hours. Outside of assignments?
- Including assignments?
- Including assignments maybe four hours.
- And, well what do you do with that time? Part of the time you're doing the assignment.
- Yeah. Mostly just going over examples and that. That would be, doing the assignment would be a good one, but I always study previous to the assignment. Going over notes and then trying to do the assignment.
- And what do you do when you go over notes?
- Do questions actually.
- Okay. Um. And if you run into a difficulty what do you do?
- Go to another question. No actually I just review again. Look in the textbook. Usually you can figure it out I think.
- Okay. When you're working with calculus problems and ideas how confident do you feel in what you're doing?
- Fairly confident. I think calculus if you get into a method of thinking it's just a process, it seems to be the same sort of process and you just get into that method of thinking and it's all very logical.
- How do you decide when you are doing it right?
- Through logic. I guess.
- Okay. Well if you get an answer and a book or somewhere gets a different answer how do you decide from there?
- Well you use, either I review what I've done. Where I could have gone wrong. And it seems, well I usually probably go wrong in some place. But ah, I would say (unclear) what I'm doing because the steps you take to achieve this answer are just so straight forward.
- Okay. What does it mean to you to understand something?
- Understand something? To take that tiny basis of logic and be able to build on it. Like using that maybe as a corner stone, but like if you understand that then you can understand things more. So if I can't understand the concepts then I know I won't be able to do anything else. But if I can understand that, I know I've understood, then you can continue onto a higher level.
- Okay. So if you were to say I understand derivatives because . . .
- By applying it to, I would understand derivatives because um (pause).
- You said something about apply.

- By applying it to another concept. Like there's. How can I say it? (pause) Through a practical application. (pause) I would understand derivatives by maybe drawing the graph or something like that. And knowing it would be right.
- Okay. We'll go onto number 11 now. When I talk about language I mean the terminology, I mean the descriptions, I mean the notation and symbols. Um. How do you find, what's used in class or in the book, in terms of helping or hindering you what are the things that confuse you and what are the things the help?
- Hm. I would definitely say the classes. The language is easier to understand. I think in the book they seem to get too theoretical. The function da da da da does not exist. Like it just seems to be like on a higher level. They don't seem to simplify it enough so I can understand it.
- And in class you find it simpler?
- Yes.
- In what way?
- Just in the way the prof would describe things in simple language. It wouldn't be like doctorate language in calculus. It's something that I can apply or I can relate to.
- Can you say more about you mean by simple language? Or give examples?
- Um. Let's see. (very long pause) Um. (pause) Hm. (pause) By simple I would say that it's something that (pause) maybe not ah (pause) it's identifiable. It's something that you can see and like right away you can see what he's talking about. And then continue from there. In the book you're kind of vague about some points, and you continue being vague. So I think if through simple you can understand the small concept then you build on that. But in the book it just seems hard. In the textbook it's just kind of, well, you're not quite sure what he's talking about.
- Alright. Um. What are the things in calculus you find easy and what are the hard things?
- Um. (pause) I find definitions of things, like ah (pause) say the definition of the derivative like f at x plus h. I find those things more difficult than the actual "give me a numerical value" type thing. I find the theory is harder than the actual practical applications.
- Okay. Um. Are there things outside the content of calculus, other things you're doing or um just the way the whole course is set up that either help or hinder your learning? It could be the class size, the format, or you're involved in drama, or something like that. Anything?
- (pause) Could it be what I've taken before?
- Does that help you?
- I think so. Yeah. It does.
- In what way?
- It's just exactly you're familiarized with the type of thinking you should be doing. More than anything like how to do a problem, and it's how you should do a problem, or how you should think about a problem. I think that has helped a lot.
- Okay.
- Physics I would say helps a lot in calculus too. It's the same sort of mind frame kind of logic.
- So you are being exposed to calculus in other courses, right?
- Partly, yeah.
- A little bit?
- No one has really taken any calculus previously. They mention it. It would be easier if we had calculus, but they ah.
- Okay. Um. You have taken Math 31, right?
- Yeah.
- How well did you do in Math 31?

- I did okay, but I'm not really sure how, like near the end I was getting a little vague about things. I did like 80 percent. Which is good, but it just seemed like I wasn't quite sure of my abilities to do it, so I (unclear).
- Okay. That's all. Unless you have anything else you want to say? Any feedback to me about the course.
- No. I think we've pretty well covered it.

Follow-up Interview

*Problem 2*

- Um. Alright this one about the two sequences where I asked what the limit would be. At that point I asked you to explain how you knew and that was something you said you just sort of knew. And I'm wondering if you have any more you could say at this point? On the first one. You knew it was zero. That was definite, but you weren't really sure how to explain it. That it is that.
- Well gee.
- If you don't have any more to explain that's okay too.
- All I can say is just these numbers. They're just decreasing without bound.
- Okay.
- And the smallest I assume would be zero.

*Problem 3a*

- Okay. Um. On number three where you went through and divided through um by the highest power term. And I asked you why you did that and you said it's just something I've been taught. Um. So do you find you have a lot of that in this course? Or not? You can go through and you can do it? And you know in this case to do such and such? But you don't know why? Does that come up a lot or not?
- Yeah. It's just, well for certain things. There's just like there's certain rules which are just rules. And you can just build from there, but I wouldn't know how to get those basic rules.
- Those basic rules are not clear to you?
- No.
- So in this case it's not clear why you would divide by the highest power term but you know you . . .
- It just makes sense to get a one and then a . . .
- Okay. And it works it?
- Yeah.

*Problem 4*

- Okay. Um. The next page. (pause) You went through first and plugged in the value two into this function. And then you went right away and factored it. And I asked you what prompted you to factor it and you said it just looks factorable. Now I'm wondering well what, did you see a purpose in factoring it even once you had done it did you see what happened?
- Well I didn't factor it. It was an undefined function. And it looks factorable. It just, like I, and it was undefined and I was trying to go from there and find if there was any other way you could make it work out. And that's just it.

*Problem 5*

- Okay. Um. Let's look at the continuity one. Um. (pause)
- This one was kind of weird.
- Let's look at. Let's turn it over so you can write. If I gave you, like that one, y equals x squared. And said prove to me it's continuous at x equals zero. Or if you want to just do it in general. Or pick a different point, you can do that. What would you do?

- I'd graph the function.
- Okay. And then what?
- And then, well ah. Couldn't you just plug it in? Like plug zero into the equation.
- And what would that tell you?
- It would tell you that the equation exists maybe. Like you could define y prime. It would be like 2 x. Whatever. And then plug in a zero for that. Then y prime would be the slope or whatever.
- Okay. And how would that tell you it was continuous?
- Ah. I don't know. Back maybe, looking back to the graph. You can tell that there's no ah hole in the graph. It's a continuous function.
- Um. If I asked you to do one that was discontinuous. Something like this one here. Discontinuous at a point. Could you prove to me symbolically, we can see it . . .
- That it's discontinuous?
- Yeah. Here's the function. In symbolic form. And here's its graph. Could you symbolically prove that it was discontinuous at one?
- Um. Well couldn't you do like the limits? Try and equate the limits. The function (unclear).
- Okay. Could you say more about that? Or write down what it is you are saying?
- Like if you have an equation like f at x is equal to whatever, so and so. And then find the limit of that equation, you can, this term there. And then find the limit again using that one. They're equal. Like equal to two.
- Say that again. They're equal . . .
- If um the function of this one equals to the function of that one. You've got continuity.
- Okay. Would that happen there?
- Ah. (long pause) No.
- Okay.
- Well (unclear) here. For x being, I don't know what the function is. For x being less than negative one. It would be just negative one right. And the second (unclear). For it being less than one, or greater than one, would be two. And these two don't equate.
- Alright. So how does the limit come into it?
- Well I. I. Well, I don't know. Wouldn't you need, like I was thinking of this equation. Like (unclear) k x is equal to x or something.
- Well, if k x is equal to negative one.
- Oh. Okay.
- In this case, and k x is equal to two.
- Mm. I don't know how at this point.

*Problem 9*

- Okay. Let's go on. (pause) (mumbling) Okay. This one here. Number nine. Places like this where there wasn't a derivative. You were able to identify lots of places. Um. Could you tell me why there isn't a derivative at something like down here? Or tha this endpoint on the left there?
- Um. (pause) Maybe here, at that point maybe there would be no slope.
- In what way? How do you mean?
- Like it goes to a point at that point. It's just an isolated point. No slope.
- Well what exactly do you mean by no slope there?
- Well if there's a point like that. Like a point there. It's a point which has a slope. But if it's just sort of like maybe there, that point there wouldn't have a slope.
- Okay. And what's that got to do with the derivative?
- Well. Isn't the derivative of a function the slope of a function?
- Okay. And that's how you would decide for something like that?
- Sure (laughing).
- Are you unsure about it?
- Yeah. I am.

- But you sense that it's kind of right? But you're not sure.
- Yeah. Part of it.
- Alright. Would you be able to prove it algebraically or symbolically? That there was no derivative at say this point here?
- Ah. (long pause) Maybe if you were given a function. Plugging in the values.
- In what way? If I give you an equation, what would you do with that equation?
- Oh. I'd try and find the slope.
- How about if I gave you something like this? (pause) We won't give that particular one, but um this one. The absolute value. Could you show me that at zero, x equals zero, there isn't a derivative?
- (pause) Probably not.
- Do you have any idea where you'd start? Using the symbolic form of it?
- Maybe, well at x is less than zero. Or x is greater than zero.
- And what would you do?
- Then ah. Plug in like x is greater than zero, so one. Plug it back into there.
- And what are you trying to show there?
- Um. And then um. Well gee.
- I want you to prove there's no derivative.
- Yup. Maybe you have to, you probably have to find the derivative of this equation. So the derivative of just x. Okay. (unclear) (pause) Oh okay and then to find the derivative.
- I want you to show there isn't a derivative.
- Yes, but would you find the derivative and then using these values plug them back into like the just plain function?

*Problem 11b*

- Let's look at part b here. Just read part b again and tell me how you read that sentence. How do you interpret it? What does it mean to you? Don't worry about what the answer is or isn't. Just read part b and tell me what it means to you.
- (pause) Slope is decreasing.
- Okay. And what does that mean? (pause) When the slope is decreasing, what would be going on? That's how you read it, right?
- Yup. Ah. (long pause) Well with the rate like y prime with respect to t. The rate is decreasing. Ah. The derivative is decreasing? I think.
- Okay. You said derivative is equal to the rate, right?
- Then when the rate is decreasing, the derivative is getting small.

*Problem 12*

- (pause) Um. I don't know. You might want to take another quick look at this one. You were struggling with it.
- Yeah.
- And ah see if there's anything more you are able to do with it. Maybe you just weren't thinking clear. 'Cause it was near the end, so it was tiring.
- (long pause) What I find most, I like being given an equation. I don't know if I could figure it out without an equation.
- Any idea why that is? What is it that . . .
- I don't know. It just seems like it's a base that you can work from. And everything seems to rely on it. Like plugging values back into it. The points of inflection and critical points and stuff like that.
- Okay. So when I don't give you the equation, I give you some sort of description, you don't know what to do to start it?
- Yeah. Yeah. I wouldn't know how to connect them all together.

Personal Interview

- Okay. Um. The other questions I have, these are all on the more general things. Um. And I'm just going to in many cases throw back at you words that you gave me. First of all you said that you didn't like the textbook because you found it too theoretical. It wasn't very practical. It didn't use enough common words. Can you say more about that? Your sort of impressions of the book, how you make use of it, and how you make use of it versus using the lecture notes, and that sort of thing.
- Well um I found with the textbook that they use like everything is related back into terms of limits. But they take like very complicated explanations, like lots of times they don't make it simple again. Like they just keep on a higher level. And I think it's easier if when they explain something in the higher level, but you have simple example just to show, like in our lectures. (unclear) for an example.
- Okay. Um. Do you make much use of the textbook?
- No.
- You don't read through it very often?
- No, I just use my notes usually.
- And what do you do with your notes in terms of using them?
- Um. Like, for the assignment or something I just look at the examples. And just relate them to what (unclear).
- Okay. When you say relate you mean you take the example and you've got your exercise and you just sort of copy the same pattern? Or do you try to figure out why he did the steps?
- Yeah. Well I think usually the examples (unclear) obvious. Usually it's like an example is just like a supplementary thing. Like what do I do next. Well, okay well let's see. So I can go from there if I get stuck. (unclear)
- Okay. Um. What use do you see the labs? You did say you found them very simple. But what value are they to you?
- I think they give you a lot of confidence. Like working with it. You're not sort of intimidated by things. So you get confidence working with the type of things, and I think with calculus it's sort of a mode of thinking that you kind of have to get into. And the labs just help you get into that sort of process.
- Okay. Um. Then. Is it that you find working through the lab manual useful? Or going to the lab? Do you work through it in advance or what?
- Yeah I do. Um.
- What do you do? Do you work through it and do the work or what?
- Do the problems, yeah.
- Do you read through the notes ahead of time or not?
- Yeah.
- And what's your impression of those?
- I like the notes. They're really pretty well basic sort of thing. And once again they rely a lot, very heavily on the examples.
- Okay. You did say that you found, like one thing was you were given confidence by doing questions from the lab because the steps were always laid out for you. So you could go through it and progress somewhere.
- Yeah. And if you can do real simple ones, well it's easier to do the hard ones.
- Okay. Um. I also asked you how you decided right or wrong. You said that you know you use logic to do it. What all does that involve? When you say you use logic what, can you be more specific?
- Um. (pause) How would I know an answer is right? (pause) Well.
- What are some of the specific things you do?
- Check in the back of the book.
- Other than check in the back of the book?

- Um. Well I go through the examples and if it seems like if there were any steps where I would have gone wrong, I usually check back with the derivative. 'Cause that's where I make a lot of stupid little mistakes.
- You check some of those calculations?
- Yup. Yeah.
- Okay. (disruption for phone call) What was I saying? About the logic. And deciding what was right or wrong.
- Well like again, I'd go back and check my steps. And if I don't see that there's a place where I could have went wrong, I just.
- Okay. Well do you have any sense that you, you're able to fit it all together yourself? Or do you rely on sort of, well external checks somehow that somebody else gives you?
- Well like I, like I, this will sound kind of weird but I find calculus it's just a way of thinking. Then if you can establish that sort of process, then things just seem to make sense.
- Okay. Do you feel you're able to achieve that way of thinking?
- Yeah. Like, yeah.
- Do you know, can you describe that way of thinking?
- Yeah, it's really, it's kind of weird actually. Um. (pause) I wouldn't know. It's a lot like physics I think. Sort of just, just, I wouldn't know how to explain it. Like it's just there. Like when I took 31, you just seem to get into, there's like calculus mode and there's a math mode type thing. And calculus is different from math.
- How is it different?
- Well I find it's, if you can relate it on a lot broader scale. It's broader and can encompass a lot of things.
- Okay. Um. Well when you're doing exercise problems what are the sorts of things you are focusing on?
- (pause) Like. Hm. Well I think it would change with each sort of exercise.
- Can you give some examples then? If you're doing a word problem what do you do? Or if you're doing a derivative problem what are the sorts of things you focus on? What is it you're trying to get out of what you're doing?
- Well actually, maybe you try and like establish where you, like establish what you want to do. Like what you want to find. And how you would go about finding that. Like which sort of route you'd take. Like with um graphing or something like that. Graphing I think is a very sort of step by step sort of procedure. And ah. I don't know. Hopefully you're right in the end.
- Okay. Um. (mumbling) Well another thing I'd asked, and I want to get more of your ideas on it, is how you see the language, meaning the symbols, the terminology, the descriptions, and other things. How does that either help or hinder your learning? In what way is it good for you? What are the things that don't work for you? Anything at all.
- If there's like, this will sound really silly, but functions. Function is like a word that people, that bothers me. Like I'd rather like, the derivative, I'd rather say y prime. Or something like that. I don't know why that would hinder me.
- Is it just the word? You just call things something and you don't know why it is that name?
- Yeah. yeah. It's just, I don't know. Ah. Like it's, it kind of hinders, but I don't see how there would be any other way to like explain it. Like I think it's hindering but necessary.
- Okay. Um. Well when you're working with all the symbolic stuff, um, do you feel confident in what you're doing? Do you have any sense of what it's about? Or is it just . . .

- Yeah I do need an example (unclear). Like it's sometimes you can, it seems to get kind of complicated, and just like using the symbols it gets complicated. But if you just have an example, it's easier.
- Okay. Well do the symbols have any meaning to you? Either you can sort of relate them to something concrete, or you understand how they relate to each other? Or is to you just a bunch of symbols that you do things with?
- Yeah I can see how they can (tape runs out). They give you symbols. Like if they give you symbols, like an equation or theorem type thing. And then they have an example and you understand what the theorem is about, then I think you can apply it to concrete things.
- Okay. You don't see anything really blocking (unclear).
- No.
- Um. When you work through a lot of exercises what sort of happens as you do more of them? Do you find that by just doing it you're learning?
- Yeah.
- And what is it? What is it you're learning by doing it?
- Calculus. It's just a learning how to work through things. Learning how, what you need to do and when type thing. What kind of steps you should be taking to solve things.
- Well how do you decide when you understand calculus?
- Ah. Gee. Um. (pause) Well if you understand. I think if you start understanding the theorems and how they derive these theorems then that's understanding.
- Okay. Have you achieved that?
- To some extent, yes. There's certain one, you know, these are rules and okay then that's great, I'll follow these rules.
- Does that bother you or not?
- Well it would be nice to know what's going on, but sure, like it doesn't bother me too much.
- Would you say you're fairly satisfied with what level of learning you're at, or not?
- Yeah. I think so.
- And the way it has gone about? You don't feel you're doing these things but you don't see what it's about?
- No. I generally, you know (unclear).
- Okay. Um. Well what, I don't know how to word this? Um. What's sort of your sense of the course now that you're almost done? You know, do you feel like it was worthwhile and good for you?
- Yeah.
- Or is it thank God it's over and I just want to get through it?
- No actually I, I didn't mind this course that much. Um. (pause) I think maybe it could have been a lot easier if you like, it tends to intimidate people. The course. But I think if they had not let it, it would have been a lot easier.
- Okay. Um. Is there anything you could say about how you see, for your learning, in particular. Forget about anyone else and what does or doesn't work for them. How, have there been things that are good for you, and how has it helped you to learn? And how are there things that you wish were different that would have helped you to learn better?
- (long pause) Um. I don't know. Maybe the time factor. The amount of work that you put into it. It relies heavily on how much work you put into it.
- Do you feel you've been able to put sufficient time into it?
- Yeah. I think so. Yeah. I think it demands a lot more time than, like it does demand a lot of time. (unclear) keep track of it (unclear) others. Which is kind of annoying, but hey.

- Okay. Um. Do you have any sense that if somebody asked you to sort of explain to them, or you know, recreate some of the main ideas, are you able to do that? Do you try to do that ever? Do you understand what I'm saying?
- Um. (pause) I don't know really.
- When you are studying and working on calculus is it more aimed at understanding the questions and the steps, than why (unclear)?
- Yeah, I think it's more the specific things rather than the general.
- Okay. That's all I have.

**Tanya's Transcripts**

Interview 1

*Problem 1*

- (long pause) How something changes in a certain amount of time. Space. (pause) How something changes in a certain amount of time or space.
- That's what calculus is about?
- A certain, a certain part of it. You know, 'cause there's different (unclear) for different things. Um. There's a lot of also regular math, as well as calculus.
- What you're saying, is that the calculus part of it?
- Yeah. It's just the rate of change of something at a certain point in time . . .
- Okay. Anything else you want to say about it?
- If it were to this person probably not. No. I would have said that it does do with a lot of graphing. That's what I would have said. A lot of graphing.
- Okay. And how does the graphing relate to it all?
- Well if a person doesn't know how to graph they're in trouble.

*Problem 2*

- This would be zero. This one is four.
- And how did you decide these?
- Well this keeps going. This is a certain number. This is smaller than that. This is even smaller. Even smaller. Eventually you'll get one over a bunch of numbers. A bunch of zeroes. And that's so close to zero, eventually you'll get to pretty well one over infinity. And that's as close as you ever probably can get. And this keeps going. This, obviously nine's are just going to keep on increasing. And it's going to get closer and closer to four.
- Will it ever reach four?
- No. And this will never reach zero.
- Um. If I asked you to write algebraically and convince me that it gets very close to four, could you do that?
- Algebraically? (pause) I'd say ah that. Um. (pause)
- And what does that represent?
- (long pause) (writing heard) Any infinitesimal.
- Okay.
- See but here you're starting off with three's, and I've started off with a zero, so I'm not sure if that would be all that correct.
- Um. But you're saying here, what is it you're saying by four minus dx? What does that represent?
- Four minus an infinitesimal amount. Which isn't really four, but it's as close to four as you get, which is kind of like this there.
- What I'm going to do is take a factor of M to the fourth out of every one of them. (pause) I take it that M is an infinite.
- Yes.
- (long wait) I can't do simple algebra today. (erasing heard) (writing)
- Okay. Now can you tell me what you've done? You took out a factor of four from every term.
- Right.
- Can you tell me why you did that?
- Because this is, this would be an indeterminate amount. Ah. You can't really see what's happening because this is infinity over infinity. And that really doesn't say anything. I'm not sure if my process was right. I did that kind of quick. So I took a factor of M to the fourth. I could have taken a factor of M to the third, but I probably would have had to simplify one more time. Uh. I did M to the fourth 'cause it's the

largest factor in there to just simplify it. Um. I simplified it and then just rounded off. Once again dx stands for an infinitesimal. Any finite over an infinite is an infinitesimal.

*Problem 3b*

- (long pause) I don't know if you want me to write it down or just say it out loud.
- Both. Whatever you feel comfortable with. (other instructions)
- Okay. I wasn't sure if you would go through this.
- I do keep these, but I also have the written, the verbal record too.
- The answer is the number closest to the question.
- Can you say more about that?
- Um. The number ah closest in relationship. You have the question number and then the answer number. Okay. Do you see what I'm saying? You've got the question like this. That's what this would be, the question number.
- One plus dx over dx?
- Right.
- And this would be the answer number.
- Alright.
- Before you round off these two numbers ah there shouldn't be, these two numbers should have the smallest difference. Rather than if I said this. Something like that.
- Right. Rather than saying infinity plus dx. Okay.
- These two have, like these should have the smallest difference between them that can possibly exist between them. I guess that's the best way I can say it.
- So why is it you round off?
- It's better to work with it. It's better to work with that number, and for all practical purposes it is kind of really that number. Like that sequence we had before. We just said that was four. If you're going to work with it. If you're going to use it in a function. If you're going to use it using other numbers you don't want to use three point fifty nines. You'll just say four 'cause it kind of really is four. Almost.
- Um. Is it possible to round off and have the same thing you had before rounding off?
- No. You're never going to, rounding off does not mean equal to. It will make a small difference even though it's a very tiny difference it will make a difference.
- Will it always make a small difference? Or is it possible that there is no difference?
- No. There will always be a difference or it's not a round off operation.
- Okay. If I asked you to take um, not one of these, I'll give you a different one. Asked you to round off that. Five plus three divided by two. What does it round off to?
- It doesn't round off to. It equals to four.
- Okay. Could I round it off to anything?
- No. It equals to four.
- Okay. So you see the round off is different from equals?
- Oh yeah. If this was a fraction, if this was like a decimal then you can round off.
- Well then what exactly is the difference? Between the round off and the equals?
- It's the number most closest to it once again. See with infinitesimals it's different than decimals. Because it was with decimals ah it can round off to different numbers. It can round off to different decimals. You can either round off to a rational or a real number. So it's a totally different story with decimals.
- Alright.
- I just said that if that was a decimal you can probably round off.

*Problem 4*

- It doesn't exist at x is equal to 2.
- How did you decide that?
- I found the value just looking at it. If you plug in x right there um you'll have this function over zero. And anything, anything over zero ah does not exist. I never really

found out why. Ah, but I know it doesn't exist. Ah. In Math 20 my teacher said anything over zero is undefined and he never really said why.

- Does that bother you that you don't know why?
- Yes. Yeah, it does. I know on the graph it would be an asymptote. I know that zero over something is zero. But I always thought that something over zero would be zero. And I never found out why it doesn't exist other than it's something I know.
- Why do you think it would be zero? Something over zero?
- Because it works the other way round. If it's zero over any number it's zero. Why can't a real number over zero also be zero? That's what I at first thought.
- Ah. Then you say you don't like the fact you really don't know it. Somebody told you it was that. Um. How do you feel about that? Do you like to, in this calculus course in particular, know where things come from, and try to figure it out for your own?
- Yes. Yup. I'm not all that interested in proofs. Proofs bug me.
- Why do proofs bug you?
- Especially this textbook. Um. When he does one on the board. When he does it it kind of fascinates me and I go wow you know. Somebody came up with this. But it's going through it that bothers me. It exists. Ah. It wouldn't be right now if it wasn't right. So let's just work with it. We don't need to know how some blow-joe came up with it.
- Okay. That's fine. I want you to be honest. Um. So do you feel then that when someone just tells you like this Math 20 teacher um how does that relate to your learning?
- Well um. I'm like um when he, well he was a very good teacher, but this is the only one thing in math ever that I don't know why that something over zero does not exist. Or is undefined. Well what does undefined mean? You know. Um. A number over zero I can see doesn't make all that sense, but why not? You know there's other weirder things out there that make sense, so.
- Okay. Um. What would the graph look like then around that point? You said something about an asymptote I think.
- Um.
- You say it's undefined there. What might it look like there?
- Well actually I'm seeing right now that you can factor this out and cross out the two. So I've made a fool of myself. But ah, um at 2 it would be an asymptote. And anything approaching it, if this was too. (unclear) horizontal asymptote. (unclear) Anything approaching it would never touch it. It would never touch it. It would just come close to it.
- What were you saying about factoring?
- Um. Let's see here. (pause) That's what I meant. So when x is 2 it would be negative one. But just looking at it right there I would have said it was undefined.
- So now are you saying it is negative one?
- (pause) See just, I didn't really consider the whole thing. I just said x at 2 well undefined. But I see now that you could have simplified it.
- And so now why, you're saying x. What are you saying?
- I'm saying now I didn't look at the fact that you could have simplified the top. I guess it's because in the lecture yesterday we were doing a whole bunch of these undefined things.
- So you were in already in that frame of mind?
- Right. So um now I'm just, in, with limits and with working with decimals I had always learned to simplify. And I didn't even look at that and I thought, "Oh God, I could have simplified that."
- So now that you have done this what would you say the graph looks like?
- Oh, y is equal to negative one? Um. (pause) As far as I know this would be a point as such. I wouldn't be able to graph that right now. Um. 'Cause this is a function by itself. See I'm not sure if this is a function by itself. If this was a function by itself it would not exist at 2. But having it simplified, it's just a point.

- Okay. Do you have any way of connecting that? What you just said? It's this and when you simplify it you get that?
- Um. (long pause) Actually I can't. Because I'm looking at this and without it being simplified if someone told me to graph it I'd graph it and it wouldn't exist at 2. And, but having it simplified it is a point at 2 negative one. It would be a point down here. To connect the two ideas, no, I haven't the slightest.

*Problem 5*

- (pause) This is continuous for all x. (mumbling) And this is not continuous for that. Obviously. 'Cause it can never touch it. These individual ones are continuous but this is one whole function. Ah. And it is not continuous in here.
- And why is that?
- Because the graph never touches it and in order to be continuous um ah any small change in x has to correspond with a small change in y.
- And how does that not happen here?
- Ah. A small change in x would be maybe right over here. Or right over here. And the one next to it, well I don't see a y down here. It will never touch, it will never touch the y axis. So if I have x at zero it doesn't exist. So it will never touch the y axis. This graph does not connect with this one. If I wanted um a value. If I wanted a value here, I have one here. I have it kind of there. I don't have one right here. And then these two will never join up.
- The first one you said is continuous. Can you tell me why you decided that?
- Well I know that from the x's. It's that. I know it, but these two will always keep going. And for every, for every small change in x you will always have y. Always. No matter what.
- Can you write that down in any way? Algebraically, what you're saying about the small change in x and there being a y? In other words could you take this, we'll use this one because it's not a difficult equation, and prove to me that y equals x squared is continuous?
- (writing) That I know from the course. Period.
- Okay. What is this you've written down?
- That the function x at ah plus a small amount will round off because of the small amount it has to round off to the y component of it.
- And how does it work for this particular function? For x squared?
- If I had. Okay. You want me to write down?
- Yeah.
- (writing)
- What is it you're saying there?
- Um. Kind of the same thing I was here. That any, I'll kind of do it this way. y at x. And these two x's are the same. Ah. If you take any x point and go a little bit to the left or a little bit to the right an infinitesimal amount it will round off to y at that x on the y axis.
- That's what you're saying here?
- Yup.
- Alright. What about this third graph? Is it continuous?
- Yes.
- Why?
- Well, it joins up everywhere. It, the normal English definition of continuous is you don't have to lift your pen off the paper. And you can see here you don't. And once again this definition applies to this too.
- Okay. How would you deal with this definition right here where it does join? In other words if I asked you to prove it's continuous at zero, could you do that algebraically?
- (pause) Well um. I would have to look at the point left, and a point to the right of zero. Right at zero there is a point and it's one. Right at x equal zero there's a point. Um.

And sure they're two different functions right in here. Algebraically? (writing) (mumbling)

- And how does that prove to me it's continuous at zero?
- This is a function by itself. Even though it is a split function it's still a function. Okay. You can consider it as the same function. Because it connects right at zero. You can see by the signs. This is ah a less than and equal to. This is a less than. Here it connects with the same idea I was saying right here. Take an infinitesimal point right to the left. It will round off to the function itself at x. Or, yeah, okay. And the point right left of zero to the point, to the function, which is one. I'm not sure if I'm following it myself. That's probably the best I can do algebraically.
- Okay. What about this last graph?
- No. It's not continuous.
- Why not?
- Um. (pause) 'Cause this isn't a point. I don't know. If ah. (pause) Okay. Right at one it's not continuous. If I take, if I go infinitesimally to the right of one I will go right up here. If I go infinitesimally to the left of one I will go right down here. Um. One plus dx should round off to this one, but it rounds off to this one. And ah, no, that's not right. (pause) Do you see what I'm saying though?
- Could you, well leave it there.
- If you go infinitesimally left, or right to one you'll go right up here. And if you go infinitesimally to the left of one you'll go right down here. But these two points should have an infinitesimal distance between one another in order to be continuous. And they don't. They have a finite distance.
- And what is you were writing down here? One plus dx? What is that?
- Well one plus dx would have been to the right.
- And what does it round off to then?
- Well it should round off to um (pause). Maybe what I want to say is ah these two points, infinitesimally here and infinitesimally here should round off to the same thing. Maybe that's what I'm trying to say.
- Okay. Are you able to write that down? Or that's where you're getting stuck, right?
- That's probably where I'm getting stuck. And probably just because it's been two units ago. And it's hard to remember it.

*Problem 6*

- Okay. If they've finished high school they know a lot about graphing. They know about slopes and they know about curves. All I would say to them is how the slope changes in a particular curve. I can say okay, they know that, by this time they know that you can take the slope of a straight line very easily. We cannot take the slope of a curve. So I would tell that the derivative is the slope of a curve at any particular point. That's all.
- And if I were your friend, and asked what does that mean. Can you explain to me what the slope of a curve is? Or show me in some way?
- (pause) Curve. You take the slope of a straight line by going rise over run. Okay. Rise over run. Change in y over the change in x. Here you can't do that because it keeps changing. It's not a straight line. The line keeps changing. The curve of it keeps changing. Slope always changing. So what we want to do is ah keep track of, or want to find out what the slope is doing at any particular point since you cannot take the slope over a certain amount. You cannot take the slope between this x and this x. Because the line changes. It's a curve. It's not straight. So all the derivative is it's just one particular point and we know what the slope is there. Take one particular point here we know what the slope is there.
- And how do I get it at this particular point?
- It's, do you want me to explain it?
- Yup. If you can quickly explain. How do I get the slope at a particular point?

- This is an upside down parabola. This is a function of this particular (pause). Of this particular shape. See all that, here I'd be explaining the power rule to use. Is that what you want here?
- Well what would that be? Tell me that.
- All the power rule is is bring the exponent down and multiplying what's in front. Negative two. Leave the x and minus-ing one off the top. So for the exponent you would have one up there. So it's just negative two x. Um. y is slope. 'Cause it's not the same as this y. 'Cause this y gives you the slope at any particular point. You know, whichever x you want, whichever point on this graph you want all you have to do is plug it into here and it will show, it will tell you the slope at that particular point.
- Okay. In terms of the graph if you didn't have the equation, how do I get the slope? What is the slope there?
- The slope? By taking a tangent line. A tangent line is a line, a straight line that runs, all that it does, it runs right through that point. So it's almost the same thing as having a straight line because it runs through, right through that point that you want. And all you have to do is know how to take the slope of a straight line. It's just to take the slope of this line. And it would be the same as the slope right here. You wouldn't have to take the derivative.
- Okay. Um. (pause) If I asked you to write down the formal algebraic definition of the derivative, can you do that?
- Of the derivative, or acquiring it? I know the definition. I know how to acquire it. I know all the proofs. I'm not sure what you want.
- What do you mean by acquire it then?
- How to get it. How to get the derivative.
- Okay. Can you do that, the general form for that?
- For this one?
- In general. I think we're having a communication problem.
- Yeah.
- When I say the definition of derivative, what's that mean to you.
- The definition of derivative? A small infinitesimal change in x. Over ah um an infinitesimal change in y over an infinitesimal change in x. That's what the definition of slope would be. Of, oh sorry. The definition of a derivative is the rate of the change of a function at a particular point.
- Okay. How do you write that down algebraically?
- (pause) Well, I would. He writes this. It's not something I could come up with myself. That's what he writes algebraically.
- And what does, how does that rate to the derivative?
- This would be the derivative at any point that you pick, of a function.
- Okay. And how does that relate to a graph?
- Ah. A graph. You can pick a point. Any x point on any graph. It has to be continuous in order for this to work. Ah. Plug the x in here. Plug the x in here. And work with the infinitesimals to find out what the derivative was at that x.
- Okay. You said the derivative was an infinitesimal change in y over an infinitesimal change in x.
- Umhm.
- How does that relate to this?
- Ah, dx to me is an infinitesimal change in x. This here shows an infinitesimal change in y because f at x is y.
- Alright.
- Okay. So here I'm saying y plus a little bit minus y itself. You're going a little bit of change in x over, a little bit of change in y over a little bit of change in x. And this y has to cancel out with the first one.
- Okay. Um. Can you give me examples of where derivatives are used?
- In real life except here?

- Yup.
- Sure. Ah. Going from distance to velocity in a moving object. Going from velocity to acceleration or backwards. Or backwards again. If you take the double derivative of distance you've got acceleration. I'm sure there's a lot more applications in physics than that. Ah. That's the only one I can think of.

*Problem 7*

- Yes.
- Okay. Ah. This is the product rule. We're doing the derivative of this times this. Plus the derivative of this times this. Derivative of this is ah the derivative outside the function, which brings the ten down and like doing a power rule with the brackets. Times. This is the chain rule. Times ah the derivative of what's inside. So that takes care of the derivative of this. And times this part. Plus, as I said, the derivative of this is once again doing the product rule with the brackets. Times the chain rule in here. I'm not quite sure I got the fraction part right. Ah. Times this once again.
- Okay. This bit about the outside and the inside, um, can you say more about that? Like what is it representing in any way? Is it to you just an outside and inside, and you know to do that? Or where does it come from?
- He didn't take up the rule. He didn't take up the ah the ah the proof of it. That's what I want to say. He didn't take up the proof of it but I can see what I'm doing. Ah. I know what to do when it comes to the chain rule. But outside of that I'm not sure what to do. It's a very complicated ah you know like it's graph would be pretty complicated looking I'm pretty sure. So ah the derivative to it is even more complicated. I don't want to question it. So. It's just that I do know what to do, but the proof for it, as I said before I'm not interested in proofs anyways. So I couldn't tell you why I do that.
- Okay. Um. Well what role do you see, whether it's this or anything else in your calculus class. What role do you see the proofs playing in learning? Are they important to your learning?
- Some of the simpler proofs, yes, I find them fascinating. Like just the power rule, simple power rule, um, product rule, quotient rule I find are great because I find that somebody didn't just ah wake up and write this down and I have to study it now. Ah. They actually did think it through. They actually did come up with it. But something like this I wouldn't really be interested in, as to how they came up with it. I'm just going to take their word for it.
- Okay. Why is it you wouldn't be interested in it?
- Um.
- Do you feel it doesn't help to learn it? Or what?
- No. Not that, because it's a simple process. Taking the derivative of anything is a simple process. I'm satisfied with just knowing where they got the basics from. Um. The definition of the power rule I found most fascinating. Like doing with the binomial formula. I thought that was really interesting. But ah now when you get to the higher, higher steps like this with the chain rule, I think "Okay, well, I've seen all the other proofs. I know it works. I'll just know this."
- Do you feel, I sense you feel satisfied with that. Having these rules and knowing they work even if you can't get them for yourself. And that's okay with you?
- Yup.
- Do you have a feeling you need to convince yourself.
- Well I could if I wanted to. I could take, I could ah take this part of it and plugging in, plug it into the definition I just showed you. F at x plus dx. I'm sure I could do that and get the same thing I did here. You know. If I really wanted to go and waste of a Saturday I'm sure I could.
- Okay. So would it be fair to say because you could if you had to?
- Yeah.
- But you'd rather not, so you're willing to accept it?

- Yeah. 'Cause like I know all these other proofs. I know all of them. The chain rule is all complicated, so why bother if I know the other ones.
- Okay. If I asked you to simplify this, could you do that?
- (pause) It would take a long time. But, yeah, I could do it.
- What would you do?

- I would do this with the binomial formula. Does that work? Yeah. No. (pause) The binomial formula only works when there's two numbers to an exponent, I believe. No, it doesn't. I'm confused now.
- But you feel you could simplify this?
- Yeah. Yeah.
- You could do something with these terms?
- Yeah.
- Could you do that very easily?
- Yes, but it would take a long time for sure.

*Problem 9*

- (pause) There's a derivative everywhere. (pause) Here it's zero. Here it's infinity. Um. Oh, well here's a point that wouldn't. This one wouldn't.
- Okay. Let's go back. Along here, between a half and two approximately, you said the derivative was zero.
- Right.
- And then the points at one half. What did you say there?
- It's infinity.
- Why is that? Can you explain that?
- Ah. The derivative, the derivative again is an infinitesimal change in x. If I take the point right here. And a point just to the right of it. Going from here right up to here. And that is positive infinity. It's a straight line. The derivative of a straight vertical line.
- What if you went a little to the left?
- Then that would be, then that wouldn't exist. Right at these points it wouldn't exist.
- Okay. So right at this point it wouldn't exist? And how does the infinity relate to it?
- Oh. Right at this point here doing ah a left-hand derivative it wouldn't exist, and a right-hand derivative it would. And what was your question?
- The left-hand derivative you say wouldn't exist.
- It wouldn't exist.
- Why not?
- Because um (pause). I know there's an explanation for this. Right at this point if you magnify it. You're magnifying the point and you still have a straight line. In order to have a derivative you need a line. You don't need a point and a line to the left or the right of it. You need ah a line where you can draw a tangent line and a slope to it. Here, like I said, a derivative just to the right of it exists. Left. Sorry. Just to the left it exists. Infinitesimally. Right at that point it doesn't exist.
- Okay. Just to the left it exists. And what would it be?
- Zero.
- Okay. And right at that it wouldn't exist.
- It wouldn't exist.
- And what about just to the right?
- Just to the right it would be infinity.
- Okay. That one you did explain. Um. Where else?
- Well once again right at this point it wouldn't exist. But right to the right of it it would. Um. (pause) Right at this point it does not exist. But infinitesimally to the right and infinitesimally to the left it's infinity. Because right down here it's pretty close to a straight line. It will still be a curve because it's approaching the asymptote, but for all practical purposes it will be infinity. It's almost a straight line.

- Okay. Are there other places there's no derivative?
- (pause) Right here. Because as I said you need some kind of a straight line or a curve. And if you magnify this then you're still going to have a V and you can't take the derivative of just a point. And that's just a point.
- Okay. When you go through this magnification and you say you need a straight line. Can you write that algebraically? In other words, this process you've explained to me about magnifying not a V, not a piece of a line, could you algebraically convince me and say, like you could pick this one or this one down here. It doesn't matter. Pick one you want. And algebraically show me why there is no derivative.
- (pause) It's got to do with continuity. This is just a crack. (pause)
- You're going to work with one of these?
- Yeah. (long pause) I probably couldn't. If I had a little while.
- What is it you've put?
- This is continuity again. Um.
- Do you know how the derivative relates to continuity?
- (long pause) I think so. Again, this is a couple of units ago, so it's hard to remember. Um. Yes, the derivative has to be, the function has to be continuous wherever the derivative is taken.
- And is that what is prompting you to work with continuity?
- Yeah. See what's mixing me up is it's continuous here. (pause) It's not continuous there. And the derivative exists there. No. Not right there. Just a little bit to the right. It's continuous here but I know it doesn't exist there. And the teacher explained why, it's just I couldn't give it back to you.

*Problem 10*

- I took the derivative of this implicitly. Um.
- Why did you do that?
- 'Cause you are taking the derivative with respect to x I'm assuming. Or I could have taken the derivative in respect to y, but then I would have to do (unclear). All I have to do . . .
- What prompted you to take the derivative?
- Because we're talking about the tangent line to the curve. So here, of course I'm saying the slope, tangent line, right there, that tells me, 'cause for a curve you cannot take the derivative as I said. It has to be a straight line for it to take a normal slope. If it's a curve and a tangent line, derivative. Ah, do you know why I did it implicitly or . . .
- Can you tell me why you did it implicitly?
- Well as before when you take the derivative you take it in respect to one variable. And the other one you have to do it implicitly and treat it like a different function. Um.
- Can I ask, what's this y prime mean to you then? Is it just a symbol you know to write down or what is it?
- All I know is that when I take, when I take the derivative in respect to something, other than the variable I'm taking it in respect to, it's almost like the chain rule. You take ah, you take the derivative and you treat it as if it was that variable multiplied by the derivative of itself, which would be y prime. So here I did um 2 y, which would be like the derivative of itself if it was with respect to y. Times y prime. Um. I know that, I just know that from Math 31. And I know that now. And it's like treating it like inside the function and outside the function. That's kind of how the professor explained. And then I just solved, I just solved for y prime and plugged the numbers.

*Problem 11*

- (pause) Right here. (pause) The number (mumbling). This is the highest slope on the graph.
- Okay. And how does the highest slope relate to the number of elk increasing most rapidly?

- Here it goes from one thousand to 35 hundred in oh two and half years or whatever. Ah. You won't find that anywhere else on the graph. You won't find that kind of an increase anywhere else on the graph. You won't find that it increases three, ah two and half thousand in a year and a half. You may find it close here, but that's for sure where it's highest.
- Okay. What about part e?
- (pause) (mumbling)
- Why those three points?
- 'Cause they're stationary. Right in this one year you're still at seven hundred, or whatever, four hundred. Here, between this one you're still there. Between this one here you're still there. And right up here, which is probably a month, it doesn't, it's not all that stationary up there either. You're just at two thousand You're not changing in that time.
- What is it that's not changing?
- In this year span the amount of wolves isn't changing. You're at a stationary point. A straight point. Ah. Here, in this one year, during the whole year, you're at probably about a thousand wolves. During the whole year they're not going up or down.
- What is it you mean by stationary then? A stationary point?
- In reference to wolves or what?
- Yeah.
- None of them are dying. None of them are being born. There are one thousand during the whole year. If during the whole year they went from one thousand to five thousand, they increased.
- Okay. (tape runs out)
- (waiting)
- Why did you mark those two points?
- That is where the rate of change increased. Was decreasing. Actually I'm thinking number-wise here. This is where they decreased in number also. Um. (pause) See when I hear rate of change I think slope. So I think this is probably the most negative slope you'll get on this graph. I'm not sure if it has any link with this. Just number-wise or the rate of change of numbers.

*Number 12*

- You didn't talk about minimum. This is a local not a global, else I'd be worried about that.
- Okay. I'm not worried about that. If I gave you this. (writing) Asked you to interpret that and give me a shape, a piece of a graph that has that property, those two, could you do that?
- (long pause)
- And why have you done that?
- Slope here is decreasing. Slope here is decreasing. But the rate of change of the slope is increasing because it's going upwards into a curve. It's going upwards because it's changing from a negative slope to a positive slope. So the rate of change of the slope is positive. Going from negative to positive.
- Where did you get that from? From what part here?
- Where do I get what?
- What told you to do the rate of change of the slope increasing? I've given you these two things. How is it you're interpreting this?
- Well I have to . . .
- You told me that's the graph.
- Right. Well I have to connect them. I have to get a graph that has the same properties for both of them.
- So what's this property you're interpreting here?
- Slope is the derivative.

- And what about this?
- The second derivative is positive means, second derivative is the rate of change of the first derivative.

Personal Interview

- Turn on the radio. Because math, and sometimes this is the only course that I can do ah turning on the radio I find it helps me concentrate more. And I enjoy the work that way a little bit more too. Like (unclear). It's not important to you. I don't even have to look at my notes from class. Because I follow everything he does in class and I understood it in class. All I I have to do is go home and open up the book to where the questions are. Hope I understand the guy who wrote the book, what he's trying to ask. I have a hard time with that sometimes. And then just to do the work. Um. If I don't understand the question, or if I'm a little stuck I'll look at the answer and I'll try to learn that way if I'm sure what to do. Sometimes he'll show us what to do in the back.
- What if you didn't have the answers? What would you do if you ran into a problem?
- I would think well why do I have this problem if I understood everything in class? And one conclusion would be that I don't understand the question. Or he didn't cover it in class and maybe it's a question I don't, I won't have to worry about.
- How do you decide when things are right and wrong?
- About my answer? I look in the back. There. And ah if there isn't an answer in the back I look at okay did I follow the steps correctly. And if I did I know it's going to be right. Maybe if I didn't add something right somewhere or something. The answer itself is not important to me. It's following the steps, making sure I did everything correctly. That's what's important to me.
- Okay. Do you go through your notes and read the textbook regularly?
- Um. I don't touch the textbook because he makes me angry.
- Why?
- Ah. You know the girl before me was saying almost the same thing. I gave up on him a long time ago. He uses words and illustrations which make it even harder to understand. It's almost as on purpose he tries to make it simple but at the same time he uses words and illustrations and ah subscripts that just looking at it you get scared. And that's not what you want to do. Usually when you write a textbook the author goes okay this is what you do here. Make sure you understand this. And ah, a good author, and writes like that, instead of just okay this is what it is. Boom. And here you've got a graph. Here's an example, and all these subscripts, and x's and little things that you don't understand. And it makes me feel like I don't want to read it. I understand Dr. X perfectly. He will cover what I have to know so why do I have to go here and just get mixed up.
- It seems to me as you go through these you have a fairly good grasp of all this notation. I asked many times for you to do it algebraically. And you could do it where people usually can't when I ask them.
- Well that's because Dr. X does it in class, and I understand him perfectly.
- Well what is it he does that helps you get that language aspect, the notation and words?
- Well, he talks while he does it. He talks. The author just writes it down. I'm just looking at it. That's all I'm doing. Just looking at it. There as he's talking about it I'm listening, I'm seeing, I'm hearing, and I'm interpreting. All at the same time. Looking at it, all dx will be negative. Well, big deal. You know. I don't like the author at all.
- You don't know what that is? Is that what you're saying?
- Well, I'm sure like in the book if um, in the book without having a lesson first of all, I would be scrambling, going "What is this?" Like the author might just mention in a little note what this means. But it wouldn't mean anything. He'd say it, but it wouldn't mean anything to me. That's what I don't like about the author. He will say things. He will tell you things, teach you things. And once again, the jargon and the symbols and

everything, they just get me upset. So I just wait for the lesson. And with reference to my own notes, I go through them like before a test.

- What do you do when you go through them?
- Actually, when I study for an exam is the only time I go back into my notes pretty well. And I don't even think I go through my own written notes. I do the sample exam in the book, in his book. And I do the sample exam in the library that Dr. X has. And that's pretty well all I need to do.
- So you do a lot of practice exercises?
- Yes. Oh yeah.
- Do you, either after class, well you said you don't really ever go through notes. Do you ever try to recreate the ideas for yourself to try to understand it?
- No, because I do that in class. I follow him in class. I know what he's doing. He kind of got me lost yesterday on this really bizarre, well he kind of says this is the derivative of the integral, but does it work backwards? Wait a minute. I can't even do it frontwards. So that's the kind of thing I walk out of class and start thinking about. Okay, okay, you know. He probably just did that. That rarely happens that I have to walk out of class and think about what he said.
- Alright. What does it mean to you to understand calculus?
- Not only calculus, but I have to understand everything. Right off the start. If I don't I panic. Anyone in the class can tell you that I get on their nerves because I'm always talking. I've always got my hand up. I'm always chattering. Well why is this? Why's that? Where did you get that from? And I ask right away. It's because I find that if I sit there he'll write something on the board. I'll just sit there and I'll stare at it. While fifteen minutes later he's covering something else, and I've lost it. And I don't hear what he's saying. So I'd rather ask something now, right away, and then okay, sure, that makes sense. So um to me to understand calculus is very important. I enjoy it. And I think it's fascinating. You need an imagination of it. And so um . . .
- In what way do you need an imagination for it?
- This rate of the change of certain things and all this whole business you need an imagination. You need an imagination not only on paper but you have to kind of see what happens, what's happening at a certain time. You need to see that. And yet a lot of what is going on in calculus with infinity, and infinitesimals, and adding them, and subtracting them. Sure you can do that on paper, but you kind of have to see what goes on. You kind of have to imagine that these sequences keep on going. They just don't stop. So you need an imagination for it. 'Cause you can't pick it up and go. Hm.
- How do you know when you understand it? What tells you that you understand it?
- That it makes sense. That it makes sense. Yeah, okay. It makes sense.
- Okay. Um. What, let me just, (mumbling). What are the things you find easy about calculus and what are the things you find hard?
- (pause) Things that I find hard are um hidden, hidden notations. The one thing. Can I give you an example?
- Sure.
- Ah. Summation notation. When you're not starting off with one right away. It's hidden notation. It interpreted. You've got a sign up here above the sigma. You've got two signs down there, and you've got a function. You really have to, I don't find it hard, but you have to think about it until you're used to it. Sigma notation and certain other things have kind of always been my weakness. But other than that everything is easy.
- So it's related to the notation and interpreting it?
- Yup. Yeah.
- And are there things you find fairly easy?
- Everything.
- Okay. Um. What are the things you find help and hinder your learning that are outside of calculus. The content. Other things in school, and other things that take away from time or give you opportunities to learn calculus? Am I making sense?

- Oh. What hinders my learning in calculus? Outside of calculus?
- Yeah.
- Nothing.
- Do you feel you have sufficient time?
- Oh yeah.
- Are you exposed to calculus in any of your other courses?
- Very little, maybe in some of my physics labs. To take time, errors, things like that. To take errors. Nothing much.
- Do you feel confident in what you're doing?
- In calculus?
- Yes.
- Oh yeah.
- And why is that?
- 'Cause I do it right. So when I do it right the second time I'll be right too, or should be.
- Alright. Um. How much time do you spend studying calculus in an average week?
- Um. (pause) Four to five hours.
- And what do you do during that time? Is it just the exercises?
- Just the homework, yup.
- Okay. Um.
- When there is an exam I study about four or five hours for the exam itself.
- And how well are you doing in the course?
- About a nine.
- Before you started the course were you apprehensive at all?
- No, because I had Math 31. I wasn't sure what to expect, and when I looked in the book before I had a lesson I panicked. Of course, right. But um um no, I had math 31, so.
- Do you see calculus as useful for your future career?
- If I want to be a math professor, yes.
- Is that what you want to do?
- Yeah.
- Do you see it as useful to society?
- Yeah. For physics and everything else. And like we were doing with the elks and the wolves. That's very important. For sure.
- Um. So I guess you're taking calculus because you want to be a math professor, so it's required, right?
- Yes.
- Why do you want to be a math professor?
- I like math. It's my strongest point. I'm not an English person. Um.
- What is it you like about math?
- Good question. Because I can, there's not a lot of reading involved. There's not a lot of expression. I have a hard time expressing, not in language, but on paper. So writing and art would be something so awful for me. I'm, I've got a good mark in English, which is surprising, but that's something I have to learn. Something I have to learn. Math's just, just comes to me and it doesn't, um, it's comprehension, spitting it out on paper. Or comprehension and working with the question. It's not as much of expressing myself.
- Okay.
- I'm not all that well with that. I can't put ideas on paper, but I can put knowledge and thoughts on paper. And that comes right through math.
- Do you see then a difference between English and math being the way it is written? Math being much more concise and precise?
- Yes. Yes.
- Do you think that's more (unclear) than more open expression?

- Yeah, I guess. The best way I can say it is it, you will write an essay and every time, no matter how perfect the essay is, somebody will find something wrong, or a different way to do it, or a different way to interpret it. Sentences can be infinitely, and paragraphs can be infinitely juggled around. And math can only be done one way. One or two ways. There's certain ways, you know. You take a step here, step two. Math can only be done one way, and that's the only one way I'll have to worry about knowing.

Follow-up Interview

- You're a person who seems very confident in doing calculus. Why is that?
- Because it stems from Math 31. I walked in not knowing anything and I found it fascinating, and I guess, I guess why I found it fascinating was because the textbook itself. When you look at it you're scared of it. It looks so complicated. But I calmed down. I went step by step. It gives you a really good feel of it to follow something that looks so complicated. And then I liked that feeling and I wanted to go on with it. And I found it fascinating because you need an imagination for the course. And then this course here is just kind of a follow-up. A review of Math 31. So that's why I like it.
- Okay. What do you mean by you need an imagination?
- Um. 'Cause you need to, you need to imagine it in your head. What goes on. You can't, you can't see infinity. You have to imagine infinity. You can't see infinitely or infinitesimally small. You have to imagine it. And on a test you don't, sometimes you don't have time to draw all the little graphs and all that's happening. You have to see in your head what's happening. With the graphing.
- Okay. And that you find very rewarding and fascinating?
- Fascinating. Yes.
- Okay. Um. Well when you had this book in Math 31 that at first looked overwhelming. Why is it that it looked so overwhelming?
- Because it's something I've never seen before. And even when you pick up um, I don't know about you but if I pick up let's say a 400 level physics book or something and looked in it it would look scary. Right. And that's how this book looked. It looked like something that wasn't, it looked like Chinese. Something I could never understand.
- Okay. But when you sat down and went carefully through it ...
- Step by step. And that's when I actually could understand.
- Okay. So this sense of achievement is a good thing for you?
- Right.
- Okay. Well then when you see that kind of thing with all this math language um that's kind of scary at first, how is it that you make it accessible to you? Like this step by step part. What's it all about? What do you do when you go step by step?
- Um. They'll say a fact and I make sure I understand it. This thing that I'm talking about in the book was derivatives from first principles. And they're complicated for somebody who doesn't have a clue what's going on. So um I would look at what they're saying. Look at their diagram. That makes sense. Go onto the next one. That makes sense. Go onto the next one. Okay, I understand. And just keep on going like that.
- Okay. So you'd take it, you'd take each part and try to make sense out of each part as you went along.
- Yes.
- Okay. Um. Well what would you say really motivates your way of learning? You seem to be aiming at understanding, wanting to understand things? Why is that? What has brought you to that point? Approaching your learning that way?
- Because if it comes to you, a question that doesn't deal with exactly what you took in class. If it's an application of it, then I'll be able to answer it. Do you see what I'm saying?
- Can you explain more?

- If we take up something in class and I understand it, but it's a totally different question on the exam, but it uses the theory behind what I learned in class. If I understand the theory and I understand the application of the theory I'll be able to answer this question on the exam. If I just memorize what we're doing in class and I just use photographic memory or something, that's not going to work for answering this question.
- Why not?
- Because you you can't ah ah learn by memorizing everything. Because you have to interpret it. You have to (unclear) with it and you have to understand the theory behind a certain form. The theory behind a certain something and then apply it to something else.
- Okay. Well when you're doing calculus and you're using your notes or working a problem, do you try to recreate the ideas for yourself or do you just (tape runs out). Together versus well I know for this I just do this? Or is a mixture of both, or?
- Well when it comes to the product rule or the power rule or the quotient rule, I know how to prove them, but I don't know why somebody came up. Well I guess in the proof you see how they came up with that. But um that's about it.
- Are you saying then certain things you don't worry about where it came from because it's not important to what you're doing at the moment?
- Yes. Mmhm. Mmhm.
- Okay. But then are there other times that um you feel you're trying to fit it together more?
- Mmhm. Mmhm. Mmhm.
- Can you think of any examples?
- Well from unit to unit you fit together everything that you learned before. Like limits applies to hyperreals and derivatives applies to hyperreals. Everything you learn applies to infinitesimals and infinites. It all fits together.
- How do you decide when you understand things?
- If it makes sense.
- In what way? How does it make sense?
- Ah. Because I can go home and do it on my own. Because I can ah, just by listening to him I can go home and apply it on my own.
- Okay. When you work through problems, your exercise questions, um, what happens as you work through more of them? What, as you get the experience doing more, what sorts of things go on?
- That ah I know this. And I feel that I know this and I don't have to go on and do any more of it.
- Okay. Do you ever find that you start doing exercises and you really don't know what you're doing but by doing exercises you get an understanding of it?
- Yes. Maybe not in this course but in past math courses.
- Okay. But in this course it's not so much that way?
- No. Because I don't let myself slip. If I don't understand something in class, I'll ask right away. I don't want to think "Well I'll understand it later." 'Cause I won't. Especially with his textbook I won't understand it later. Definitely not. And um if I don't understand something in the exercises which is usually common. If I don't understand his answer I'll go through the notes and ask what am I doing here? What am I doing?
- Okay. Um. Let's take a look here. This is the one.
- Straight line. I realized it afterwards.
- (talk about what was done in the tutorial that morning)
- But it's still undefined at two though.
- Right. It's undefined. But it's not an asymptote.
- It's zero.
- It's just a straight line and it's got a hole in it. (etc.)
- But the equation of all this graph, is it this? Or is it y equals x minus three?

- (answer) When people in general talk about "Oh I know, I just can't remember." What's that all about for you?
- Well here I was really concentrating on the ah on the interview itself. If this was on a test or something I would give it a second shot. And I realized when I had walked out that I'd made a complete fool of myself. And ah usually on a test I'm not the type of person to ah to "Oh God, (unclear)." Before the test I put in about four or five hours. Here I was (unclear) on you and what I could give you, rather than what (unclear). So on an exam I don't usually blank out unless, in a mathematical situation I don't (unclear).
- You're one of the few that when I asked them to do it symbolically could do something. Do you have any idea why that is? Why, do you find the symbols have meaning for you that way?
- Well once again, when he writes a symbol on the board I understand it. And I know when he says the definition of a derivative, even though it looks funny it's an infinitesimal change in y over an infinitesimal change in x. Even though there are x's and dx's and y's all over the place. I can see that it's a change over a change.
- Okay. How do you make use of those symbols in calculus, in the learning of calculus? Do you find they actually help you? Do you see a use for them or do you think that it's the way it's written down so you have to learn it?
- Well, continuity, it makes sense. It makes sense that in order for something to be continuous, if you look infinitesimally left of the point it should still round off to the point of that function. And if it doesn't that means that that point is somewhere else.
- Okay. Do you find that what you say there in the English you can relate it to the symbols? You have a way of translating it back and forth.
- Umhm. Some of them I couldn't. (unclear)
- But in general do you think that you are able to do that?
- I'm only able to do that with something that I have already learned. I would have had a hard time to put something symbolically or algebraically that I didn't learn how to do. Like this is stuff we took in class.
- Okay. But something that you didn't understand you would then not be able to translate it?
- No.

## Appendix U - Summary of the Analysis of the Interviews for Students' Sources of Conviction

Each interview transcript was read several times. Initially, particular attention was paid to a student's statements and comments while responding to the calculus problems in the clinical portion of the the interview. Whenever the student made reference to things learned or remembered from class, the teacher or the textbook, mathematics rules, mathematics concepts, visually or physically oriented descriptions or interpretations, or personal beliefs about mathematics concepts or procedures, the margin of the corresponding text was marked with one of the initial sources of conviction codes: IM, IE, ER, EO (see Chapter 3). Next, the entire interview transcript, and in particular the text from the personal interview, was read again. Whenever the student made reference to her or his strategies for learning calculus, personal sense of understanding of calculus, perceptions of calculus, or particular experiences in calculus, the corresponding text was underlined.

With the aid of a word processor the underlined portions of each transcript were then copied verbatim to a separate file. Through this process of re-copying a student's words, re-reading the words, and remembering the sound of the student's voice when the interview was conducted, the researcher was able to begin to identify similarities and differences in various students' sources of conviction. It was at this point that the potential of three possible types of learners was conceived. Preliminary descriptions of these three types were then formulated. Next, the individual files with interview excerpts were read several times. The original texts were often referred during these readings so as to hear students' words in the context in which they had been stated. The purpose of these intensive readings was to allow a student's own perceptions of his or her calculus learning to emerge. This overall manner of proceeding was similar to that used by Belenky et al. (1986) to analyze interviews conducted "to explore with women their experience and problems as learners and knowers" (p.11).

Finally, salient features and quotes form each interview transcript were recorded on cards, one card for each student. These cards were then studied and sorted into one of the three initially identified and described learner types. Not all students could initially be classified as members of one of these three groups. In particular, the unabridged text from the interviews with Betty, Cindy, Leanne and Nadine had to be re-examined before the researcher could confidently classify these students. The final descriptions of the three groups who differed as to the nature of their sources of conviction, as well as the names for the three groups (Collectors, Technicians and Connectors) were then developed.

## Appendix V - Reader Access to Instructional Materials for the Three Calculus Courses

For reasons of confidentiality the post-secondary institutions which formed the research settings for this study cannot be named. However, readers who are interested in learning more about the instructional materials at any of these three institutions should contact the author through one of the addresses listed below. Please state in writing your intentions in having access to these instructional materials. I will contact the appropriate individuals and forward your letter on your behalf. At the time of writing this note I am no longer residing in Canada, so please be patient if correspondence seems to be long in coming.

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